Whittaker vectors, a matrix calculus, and generalized hypergeometric functions

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- **Theorem** (Matumoto, 1987). Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded semisimple Lie algebra. Fix $s \in \mathbb{Z}_{>0}$, s.t. $\mathfrak{g}_s \neq 0$. Let $\mathfrak{n}_s = \sum_{i \geq s} \mathfrak{g}_i$, a nilpotent subalgebra of $\mathfrak{g}$. Choose $\psi \in (\mathfrak{g}_s)^*$, let $\widetilde{\psi}$ be its trivial extension to a character for $\mathfrak{n}_s$. Let M be a $U(\mathfrak{g})$ module, and put
$Wh^{alg}_{\mathfrak{n}_s,\psi}(M) =$
$\{w \in M^* \mid w(X \cdot v) = \psi(X)\, w(v) \;\; \forall\; X \in \mathfrak{n}\,,\;\; v \in M\}$.
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- θ : Cartan involution, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ corresponding Cartan decomposition of $\mathfrak{g} = Lie_{\mathbb{R}}(G)$
- Fix a nilpotent element $e \in \mathfrak{g}$, a corresponding θ-stable S-triple $\{e, h, f\}$, and a corresponding decomposition

$$\mathfrak{g} = \sum_i \mathfrak{g}_i \quad , \quad [h, Z] = iZ \quad \forall\, Z \in \mathfrak{g}_i$$

$$\mathfrak{n} = \sum_{i>0} \mathfrak{g}_i \quad , \quad \mathfrak{l} = \mathfrak{g}_0 \quad , \quad \overline{\mathfrak{n}} = \sum_{i<0} \mathfrak{g}_i$$

Let $\chi_e(\cdot) = iB(f, \cdot)$ is the differential of an admissible unitary character for $N = \exp(\mathfrak{n})$.

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- $V^{-\infty}$: the continuous dual of V^∞ (w.r.t. usual Fréchet topology of V^∞)

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For each $\mathcal{O}_i$ in wave front set of (π, V) choose representative nilpotent element $e_i \in \mathcal{O}_i$ then

$$WC(\pi) \equiv \sum_i \dim \left(Wh^{\infty}_{\mathfrak{n}_{e_i}, \chi_{e_i}}(\pi) \right) \left[\overline{\mathcal{O}_i} \right]$$

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Significance: like Barbasch-Vogan conjecture (proved by Schmid-Vilonen) this conjecture lies right at a vital crossroads of the analytic, algebraic and geometric aspects of representation theory.

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([Kostant, 2000])

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$$e = it \quad , \quad h = 2t\frac{d}{dt} + 1 \quad , \quad f = i\left(t\frac{d^2}{dt^2} + \frac{d}{dt} - \frac{r^2}{4t^2}\right)$$

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- Whittaker vectors for $e \longleftrightarrow \delta$-distributions
- Whittaker vectors for $f \longleftrightarrow$ modified Bessel functions

Idea

- Explicit Hilbert space and concrete Whittaker functionals

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- use of classical special function theory to get asymptotics of f-Whittaker vectors at 0 and ∞ to prove continuity of corresponding functionals on smooth vectors

Principal Series Representation

A family of principal series representations of $GL(2n, \mathbb{R})$
(Speh, Sahi-Stein, Sahi-Kostant, et al.)

$$P = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A, B, C \in M_{n,n}(\mathbb{R}) \right\}$$

$$L = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \mid A, C \in M_{n,n}(\mathbb{R}) \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \mid B \in M_{n,n}(\mathbb{R}) \right\}$$

$$\overline{N} = \left\{ \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \mid C \in M_{n,n}(\mathbb{R}) \right\}$$

Nonunitary principal series

Levi factor

$$L = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} \varepsilon \mathbf{1} & 0 \\ 0 & \varepsilon \mathbf{1} \end{pmatrix} \begin{pmatrix} a\mathbf{1} & 0 \\ 0 & a^{-1}\mathbf{1} \end{pmatrix} \begin{pmatrix} z\mathbf{1} & 0 \\ 0 & z\mathbf{1} \end{pmatrix}$$

with $M_1, M_2 \in SL(n, \mathbb{R})$, $\varepsilon = \pm 1$, $a, z \in \mathbb{R}_{>0}$.

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$$\begin{aligned} I(s) &= Ind_{MAN}^{G}\left(1 \otimes e^{-s\nu} \otimes 1\right) \\ &= \left\{\varphi \in C^{\infty}(G) \mid \varphi(gman) = e^{-\left(s+n^2\right)\nu(\log(a))}\varphi(g)\right\} \end{aligned}$$

Noncompact picture:

$$I\left(s\right) \approx C^{\infty}\left(\overline{\mathfrak{n}}\right) \approx M_{n,n}(\mathbb{R}) \approx \mathbb{R}^{n^2}$$

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$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \quad \overline{N} \ni \overline{n}(Y) = \exp \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$$

$$\pi(g) f(Y) = e^{-\left(s+n^2\right) \ln|\det(D-BY)|} f\left([D - BY]^{-1} [AY - C]\right)$$

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$$\pi\left(\begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix}\right) = -\frac{\partial}{\partial y_{ij}}$$

$$\pi\left(\begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix}\right) = \sum_{k,l} y_{ki} y_{jl} \frac{\partial}{\partial y_{kl}} + \left(s + n^2\right) y_{ji}$$

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- Barchini-Sepanski-Zierau : this even makes sense for the unitarizable degenerate principal series representations corresponding to the non-open orbits.
- representation of $\mathfrak{n}$, $\overline{\mathfrak{n}}$ on $L^2(\mathcal{O}, d\mu)_{smooth}$

$$\pi\left(\begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix}\right) = i x_{ij}$$

$$\pi\left(\begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix}\right) = i \sum_{k=1}^{n} \sum_{\ell=1}^{n} x_{k\ell} \frac{\partial}{\partial x_{ik}} \frac{\partial}{\partial x_{\ell j}} - s \frac{\partial}{\partial x_{ij}}$$

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- Barchini-Zierau: smooth Whittaker functionals for $\overline{\mathfrak{n}}$ correspond to δ-functionals. (The subtle part is identifying the space of smooth vectors and that the δ-functionals are continuous linear functionals on that space.)
- **Goal:** understand the smooth Whittaker functionals for $\mathfrak{n}$ as a class of special functions: i.e. find explicit solutions of

$$\pi(X)\,\Phi = \chi(X)\,\Phi \qquad \forall\, X \in \mathfrak{n}$$

with asymptotics such that

$$f \mapsto \int_{\mathcal{O}} f \Phi d\mu_{\mathcal{O}}$$

is a continuous linear functional on the space of smooth vectors.

Choose $\chi(X) = i\lambda tr(X) = iB(f, X)$, $f = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$

$$\left(\sum_{k,\ell} \frac{\partial}{x_{\ell j}} x_{\ell k} \frac{\partial}{x_{ik}} - (s + n(n-1)) \frac{\partial}{\partial x_{ij}} - \lambda \delta_{ij} \right) \Phi = 0 \quad (1)$$

Set

$$\mathbf{X} = (x_{ij})_{\substack{1\leq i\leq n\\1\leq j\leq n}} \qquad \mathbf{D} = \left(\frac{\partial}{\partial x_{ji}}\right)_{\substack{1\leq i\leq n\\1\leq j\leq n}}$$

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- then (1) can be written more succinctly as

$$(\mathbf{DXD} - (s + n(n-1))\,\mathbf{D} - \lambda \mathbf{I})\,\Phi = 0$$

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- particularly natural looking generalization of the confluent hypergeometric equation ($n = 1$).

Remark Under a change of coordinates corresponding to conjugation by $\mathbf{A} \in GL(n)$

$$(x_{ij}) \to (x'_{ij}) \equiv \left(\sum_{k,\ell} A_{ik} x'_{k\ell} A^{-1}_{\ell j} \right)$$

one has

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{X}'\mathbf{A}$$

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and so the system of PDEs (1) is actually conjugacy invariant.

Digression: a matrix calculus

Write

$$\det\left(\mathbf{X} - t\mathbf{I}\right) = \sum_{\sigma\in\mathfrak{S}_n} sgn\left(\sigma\right) \prod_{i=1}^{n} \left(x_{i\sigma(i)} - t\delta_{i\sigma(i)}\right)$$
$$= \left(-1\right)^n t^n + p_1\left(x\right) t^{n-1} + \cdots + p_n\left(x\right)\mathbf{I}$$

where

$$p_1\left(x\right) = tr\left(\mathbf{X}\right)$$
$$p_n\left(x\right) = \det\left(\mathbf{X}\right)$$

and the intermediate $p_i\left(x\right)$ are the so-called *generalized determinants*.

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Cayley-Hamilton theorem

- We set

$$\phi_i = (-1)^{n+1}\, p_i$$

so that the Cayley-Hamilton theorem takes the form

$$\mathbf{X}^n = \phi_1 \mathbf{X}^{n-1} + \phi_2 \mathbf{X}^{n-2} + \cdots + \phi_n \mathbf{I}$$

whence

$$\begin{aligned} \mathbf{X}^{n+q} &= (\mathbf{X}^n)\, \mathbf{X}^q \\ &= \phi_1 \mathbf{X}^{n+q-1} + \phi_2 \mathbf{X}^{n+q-2} + \cdots + \phi_n \mathbf{X}^q \end{aligned}$$

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Lemma $f \in \mathbb{C}[x]^G$, $\mathbf{\Phi} \in \mathcal{F}(\mathbf{X})$, then

$$\mathbf{D}(f\mathbf{\Phi}) = (\mathbf{D}f)\mathbf{\Phi} + f(\mathbf{D}\mathbf{\Phi})$$

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- **Remark:** $\mathrm{DX} - \mathrm{XD} \neq \mathrm{I}$.

Lemma $\mathrm{XD}\phi_q = \mathbf{X}^q - \sum_{i=1}^{q-1} \phi_i \mathbf{X}^{q-i}$

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Lemma

$$\psi_i = \det \begin{bmatrix} \phi_1 & 1 & 0 & \cdots & 0 \\ -2\phi_2 & \phi_1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ (-1)^{q+1}\, q\phi_q & (-1)^q\, \phi_{q-1} & \cdots & \cdots & \phi_1 \end{bmatrix}$$

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Lemma

$$\psi_i = \det \begin{bmatrix} \phi_1 & 1 & 0 & \cdots & 0 \\ -2\phi_2 & \phi_1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ (-1)^{q+1}\, q\phi_q & (-1)^q\, \phi_{q-1} & \cdots & \cdots & \phi_1 \end{bmatrix}$$

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Lemma For any partition $\lambda = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ set

$$c(\lambda) = \frac{\left(\sum_{i=1}^n m_i\right)!}{\prod_{i=1}^n m_i!}$$

and set

$$\xi_{n,q,i} = \begin{cases} \phi_{n-i} & q = 0 \\ \sum_{\lambda \in \mathcal{P}_q} c(\lambda)\, \phi_{\lambda_1} \cdots \phi_{\lambda_k} \phi_n & i = 0 \\ \sum_{j=q-i}^{q} \sum_{\lambda \in \mathcal{P}_j} c(\lambda)\, \phi_{\lambda_1} \cdots \phi_{\lambda_k} \phi_{n-i-j+q} & i = 1, \ldots, n-1 \end{cases}$$

Then, for $q = 0, 1, \ldots, n-1$,

$$\mathbf{X}^{n+q} = \sum_{i=0}^{n-1} \xi_{n,q,i} \mathbf{X}^i$$

Back to the Whittaker PDEs

$$\left(\mathbf{XDXD} - \left(s + n\left(n-1\right)\right)\mathbf{XD} - \lambda\mathbf{X}\right)\Phi = 0$$

Look for conjugacy invariant solutions with Ansatz:

$$\Phi = \sum a_{m_1\cdots m_n}\phi_1^{m_1+r_1}\cdots\phi_n^{m_n+r_n}$$

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- Indicial equations turn out to be

$$r_n(r_n - s) = 0$$

$$0 = r_{n-1} = \cdots = r_1$$

- Can establish a total ordering of recursion relations for $a_{m_1 \ldots m_n}$ and demonstate unique formal solution with $\Phi(\mathbf{0}) = 1$.

Hypergeometric functions ${}_pF_q$

Differential Equation:

$$\left[E\left(E-b_1\right)\cdots\left(E-b_q\right)-x\left(E+a_1\right)\cdots\left(E+a_p\right)\right]\ {}_pF_q=0$$

where $E=x\dfrac{d}{dx}$.

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Solution:

$${}_pF_q\left(\begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{matrix};x\right)=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_p\right)_k\,k!}x^k$$

Generalized hypergeometric functions

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Natural matrix calculus formulation
Set

$$\mathbf{E} = \mathbf{X}\mathbf{D}$$

and consider *matrix calculus hypergeometric equations* of the form

$$\left[\mathbf{E}\left(\mathbf{E} - b_1\right) \cdots \left(\mathbf{E} - b_q\right) - \mathbf{X}\left(\mathbf{E} + a_1\right) \cdots \left(\mathbf{E} + a_p\right)\right] \ {}_pF_q = 0$$

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Example

${}_2F_1^{(d)}$, generalized Gauss hypergeometric function (Kaneko, Vilenkin-Klimyk)

$${}_2F_1^{(d)}(a,b;c,\mathbf{t}) \equiv \sum_{k=1}^{\infty} \sum_{|\lambda|=k} \frac{[a]_\lambda [b]_\lambda}{[c]_\lambda\, k!} C_\lambda^{(d)}(\mathbf{t})$$

where $\mathbf{t} \in \mathbb{R}^n$, the $C_\lambda^{(d)}(\mathbf{t})$ are (a particular normalization of) the Jack symmetric polynomials, and

$$[a]_\lambda \equiv \sum_{i=1}^{l(\lambda)} \left(a - \frac{d}{2}(i-1)\right)_{\lambda_i}$$

$(a)_k = (a)(a+1)\cdots(a+k-1)$ being the usual Pochhammer symbol.

Although the functions ${}_2F_1^{(d)}$ are defined by their series expansions, one has

Theorem ([Kaneko, 1993]) ${}_2F_1^{(d)}$ is the unique solution of

$$
\begin{aligned}
0 = {} & t_i(1-t_i)\frac{\partial^2 F}{\partial t_i^2} + \\
& \left[c - \frac{d}{2}(n-1) - \left(a+b+1-\frac{d}{2}(n-1)\right)t_i\right]\frac{\partial F}{\partial t_i} \\
& + \frac{d}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{t_i(1-t_i)}{t_i - t_j}\frac{\partial F}{\partial t_i} - \frac{d}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{t_j(1-t_j)}{t_i - t_j}\frac{\partial F}{\partial t_j} - abF
\end{aligned} \tag{4}
$$

satisfying

(i) $F(\mathbf{t})$ is a symmetric function of $t_1, \ldots, t_n$

(ii) $F(\mathbf{t})$ is analytical at the origin and $F(\mathbf{0}) = 1$.

Proposition If F is a conjugacy invariant function analytic in the $\phi_1, \ldots, \phi_n$

$$\left[(\mathbf{XD})\left(\mathbf{XD} + c' - 1\right) - \mathbf{X}\left(\mathbf{XD} + a'\right)\left(\mathbf{XD} + b'\right)\right] F \qquad (5)$$

Then (5) is equivalent to (4) when $n = 2$, $d = 2$ and

$$a' = -a$$

$$b' = -b$$

$$c' = c + 1$$

explicit connection

Identities: If

$$\Phi = \sum a_{m_1 m_2} \phi_1^{m_1} \phi_2^{m_2}$$

then

$$(\mathbf{XD})\,\Phi = \frac{\partial \Phi}{\partial \phi_1}\mathbf{X} + \phi_2 \frac{\partial \Phi}{\partial \phi_2}\mathbf{I}$$

$$\mathbf{X}\,(\mathbf{XD})\,\Phi = \frac{\partial \Phi}{\partial \phi_1}\left(\phi_1 \mathbf{X} + \phi_2 \mathbf{I}\right) + \phi_2 \frac{\partial \Phi}{\partial \phi_2}\mathbf{X}$$

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$$\mathbf{X}\,(\mathbf{XD})\,\Phi = \frac{\partial \Phi}{\partial \phi_1}(\phi_1 \mathbf{X} + \phi_2 \mathbf{I}) + \phi_2 \frac{\partial \Phi}{\partial \phi_2}\mathbf{X}$$

Interpret the t_1, t_2 as the eigenvalues of $\mathbf{X}$) and make a change of variable $\phi_1 = t_1 + t_2$, $\phi_2 = t_1 t_2$.

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