Characteristic Cycles, Multiplicities and Degrees for a Class of Small Unitary Representations
OSU Lie Groups Seminar
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1. Orbital Invariants of Admissible Representations

An example of a strong invariant of an admissible representation (by which I mean an invariant capable of splitting points in $\widehat{G}_{adm}$) is the distributional character of a representation. Let G be a reductive Lie group, $(\pi, \mathcal{H})$ an irreducible admissible representation of G and $\mathcal{H}_K$ is associated Harish-Chandra $(\mathfrak{g}, K)$-module, and let Θ_π be its distributional character

$$\Theta_\pi(f) \equiv Tr_{\mathcal{H}}\left(\int f(g)\,\pi(g)\,dg\right) \qquad , \qquad \forall\ f \in C_c^\infty(G)$$

Weaker invariants are things like the annihilator of π in $U(\mathfrak{g})$, minimal K-types, Gelfand-Kirillov dimensions.

There are also several invariants associated with certain homogeneous affine varities associated with π. These I'll describe below.

1.1. Asymptotic Expansion of Characters.

Theorem 1.1 (Barbasch-Vogan 1980)**.** *Let θ_π be the lift of Θ_π via the exponential map to a neighborhood of the identity on $\mathfrak{g} = Lie(G)$. If $f \in C_c^\infty(G)$ and $t > 0$ define*

$$f_t(X) = t^{-\dim \mathfrak{g}} f\left(t^{-1}X\right)$$

Then there is an integer r and tempered distributions $\{D_i\}_{i=-r}^{\infty}$ on $\mathfrak{g}$, such that for $f \in C_c^\infty(G)$

$$\theta_\pi(f_t) \sim \sum_{i=-r}^{\infty} t^i D_i(f)$$

as $t \to 0^+$. Furthermore, the support of $\widehat{D_i}$ is a union of nilpotent orbits in $\mathfrak{g}$.

1.2. Characteristic Cycles.

Let $(\pi, \mathcal{H})$ be an irreducible admissible representation of a reductive Lie group G, and let $\mathcal{H}_K$ be its associated Harish-Chandra $(\mathfrak{g}, K)$-module.

Theorem 1.2 (Matsumura)**.** *Let R be a commutative Noetherian ring and $M \neq 0$ a finitely generated R-module. There there exists a chain*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of submodules of M such that for each i we have $M_i/M_{i-1} \simeq R/P_i$, with P_i a prime ideal of R.

Remark 1.3 (worthy of a lemma). Let R be a commutative Noetherian ring and let Q be a minimal prime ideal of R. Suppose $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ is a filtration of a finitely generated A-module M such that $M_i/M_{i-1} = R/P_i$ Then the number of times $P_i = Q$ is independent of the choice of filtration.

Definition 1.4. *Let R be a commutative Noetherian ring and $M \neq 0$ a finitely generated R-module. Let $\{P_1, \ldots, P_r\}$ be the set of minimal ideals containing the annihilator of $Ann(M) = \{r \in R \mid rM = 0\}$. The* characterisitc cycle *of M is the formal sum*

$$Ch(M) = \sum_{i=1}^{r} m(P_i, M)\, P_i$$

where $m(P_i, M)$ is a positive integer defined as follows. Choose any finite filtration of M such that each subquotient M_j/M_{j-1} is of the form R/Q_j with Q_j a prime ideal of R. $m(P_i, M)$ is the number of times that $Q_j = P_i$.

Definition 1.5. *Suppose that X is a finitely generated Harish-Chandra module. Recall that $K_{\mathbb{C}}$ acts on X. A* ***good filtration*** *$\mathcal{F}$ of X is a (possibly infinite) increasing filtration*

$$0 = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset$$

of X, satisfying the conditions listed below. Write U_n for the n^{th} of the standard filtration of $U[\mathfrak{g}]$. By the Poincaré-Birkhoff-Witt theorem, the associated graded ring $gr(U[\mathfrak{g}])$ is naturally isomorphic to $S[\mathfrak{g}]$. The conditions on a good filtration of X are

(i) *X_m is finite-dimensional and $K_{\mathbb{C}}$-invariant.*
(ii) *The union of all the X_m is X.*
(iii) *The filtrations of X and $U[\mathfrak{g}]$ are compatible:*
$$U_n X_m \subset X_{n+m} \quad .$$
(iv) *The associated graded $(S(\mathfrak{g}), K_{\mathbb{C}})$-module $gr(X)$*
$$gr_{\mathcal{F}}(X) = \bigoplus_{n=0}^{\infty} X_n/X_{n-1}$$
is finitely generated.

The $gr(X)$ inherits compatible of the $K_{\mathbb{C}}$ and $\mathfrak{g}$-actions from X, but because the action of the Lie algebra $\mathfrak{k} = Lie(K_{\mathbb{C}})$ preserves the filtration, it follows that $S(\mathfrak{g})\,\mathfrak{k} \subset Ann(gr(X))$ and so $gr(X)$ can be regarded as a $S(\mathfrak{g}/\mathfrak{k}) \simeq S(\mathfrak{p})$-module.

Definition 1.6. *Let X be the Harish-Chandra module of an admissible representation of a reductive Lie group G. Fix a good filtration of X and let $gr(X)$ be the associated $(S())$*

1.3. **Schmid-Vilonen.** Vogan and Barbasch conjectured and Schmid and Vilonen proved that the similarilty between wave front cycles and characteristic cycles can actually be expressed as a precise equality. The leading term of the local character expansion can be thought of as a complex-linear combination of nilpotent $G_{\mathbb{R}}$ orbits, allowing us to write

$$\Theta_\pi = \sum_{\mathcal{O}_j \in WF(\pi)} b_j\,[\mathcal{O}_{,j}] + \cdots$$

Schmid and Vilonen refer to

$$WF(\pi) = \sum_{\mathcal{O}_j \in WF(\pi)} b_j\,[\mathcal{O}_{,j}]$$

as the *wave front cycle* of π and prove that this wave front cycle coincides with the characteristic cycle of Vogan.

However, this coincidence is not nearly obvious as the notation suggests. First of all the “cycles” $[\mathcal{O}]$ that appear in the characteristic cycle are $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$, while the cycles that appear in the wave front cycle are nilpotent $G_{\mathbb{R}}$-orbits in the real Lie algebra $\mathfrak{g}_o$. The basis for the characteristic cycle / wave front cycle correspondence is the following theorem of Sekiguchi

Theorem 1.7 (Kostant-Sekiguchi Correspondence)**.** *Suppose that G is a real reductive Lie group with maximal compact subgroup K. Then there is a natural one-to-one correspondence between the set of nilpotent G-orbits in the real Lie algebra $\mathfrak{g}_o$ of G and the set of $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$. Suppose that under this correspondence the orbit of $\lambda_{\mathbb{R}} \in \mathcal{N}^*$ corresponds to that of $\lambda_{\mathfrak{k}} \in \mathfrak{p}^*$ and $G_{\mathbb{C}}$ is any complex group with Lie algebra $\mathfrak{g} = (\mathfrak{g}_o)_{\mathbb{C}}$, then*

- *The $G_{\mathbb{C}}$-orbits of $i\lambda_{\mathbb{R}}$ and $\lambda_{\mathfrak{k}}$ coincide.*

- $\dim_{\mathbb{R}} G \cdot \lambda_{\mathbb{R}} = 2 \cdot \dim_{\mathbb{C}} K_{\mathbb{C}} \cdot \lambda_{\mathfrak{k}} = \dim_{\mathbb{C}} G_{\mathbb{C}} \cdot \lambda_{\mathfrak{k}}$
- *The maximal compact subgroups of the isotropy subgroups $K(\lambda_{\mathfrak{k}})$ and $G_{\mathbb{R}}(\lambda_{\mathbb{R}})$ are isomorphic.*

1.4. **Whittaker Cycles?** There is a third way of attaching numbers to nilpotent orbits. Let $\mathcal{O}$ be a real nilpotent orbit occuring in the asymptotic expansion of the character of a admissible representation. Associated to $\mathcal{O}$ is a particular parabolic subgroup $P = LN$ of G, a particular character $\chi_{\mathcal{O}}$ of N, and a generalized Gelfand-Graev representation $Ind_N^G(\chi_{\mathcal{O}})$. Set

$$\begin{aligned} w_{alg}(\pi, \mathcal{O}) &= \dim Hom_{\mathfrak{g},K}\left(V, Ind_N^G(\chi_{\mathcal{O}})\right) \\ w_{-\infty}(\pi, \mathcal{O}) &= \dim Hom_G\left(V^{\infty}, Ind_N^G(\chi_{\mathcal{O}})\right) \end{aligned}$$

be the dimensions of the space of generalized Whittaker modules of type $\mathcal{O}$, and write

$$\begin{aligned} \mathcal{WC}_{alg}(\pi) &= \sum_{\mathcal{O} \in WF(\pi)} w_{alg}(\pi, \mathcal{O})[\mathcal{O}] \\ \mathcal{WC}_{-\infty}(\pi) &= \sum_{\mathcal{O} \in WF(\pi)} w_{-\infty}(\pi, \mathcal{O})[\mathcal{O}] \end{aligned}$$

Question 1.8. *Can one relate either of these Whittaker cycles to wave-front cycles (or equivalently, to associated cycles).*

Theorem 1.9 (Matumoto)**.** *For large representations the dimension of the space algebraic Whittaker vectors coincides with the Bernstein degree of the representation.*

Theorem 1.10 (Kostant, Lynch)**.** *Let $G = KAN$ be an Iwasawa decomposition of a real reductive Lie group. The principal series representations $ind_{MAN}^G(\sigma \otimes e^{\nu} \otimes 1)$ is large, and the Bernstein degree is $\#W(\mathfrak{g}_o, \mathfrak{a}_o) \cdot \dim(\sigma)$, where $\#W(\mathfrak{g}_o, \mathfrak{a}_o)$ is the dimension of the little Weyl group.*

Theorem 1.11 (Yamashita)**.** *If G is a connected simple Lie group of Hermitian type and let π be an irreducible unitary highest weight representation of G. Then the multiplicities in the wave front cycle of π and in $\mathcal{WC}_{alg}(\pi)$ coincide.*

Theorem 1.12 (Nishiyama, Ochiai, Taniguchi, Yamashita and Kato)**.** *The associated cycle and the Bernstein degree of a large $(\mathfrak{g}, K)$-module X_{π} are given by*

$$\begin{aligned} \mathcal{AC}(\pi) &= \sum_{\mathcal{O}_{\mathbb{R}}} w_{-\infty}(\pi, \mathcal{O}_{\mathbb{R}}) \cdot \left[\overline{\mathcal{O}}\right] \\ BernsteinDeg(\pi) &= \frac{w_G}{l_G} \sum_{\mathcal{O}_{\mathbb{R}}} w_{-\infty}(\pi, \mathcal{O}_{\mathbb{R}}) \end{aligned}$$

where $\mathcal{O}_{\mathbb{R}}$ runs over the principle nilpotent orbits in $\mathfrak{g}_{\mathbb{R}}$, $\mathcal{O}$ is the $K_{\mathbb{C}}$-orbit corresponding to $\mathcal{O}_{\mathbb{R}}$ via the Kostant-Sekiguchi correspondence, l_G is the number of principal nilpotent G-orbits, and w_G is the order of the little Weyl group.

In these talks, we present the results of some exploratory calculations aimed at determining and correlating the characteristic cycles, multiplicities, and Whittaker vectors for a special class of small unitary representations. Although our purpose is purely representation-theoretical, and empirical at that, it also illustrates how representation-theoretical methods, in a sufficiently equivariant setting, might be used to answer rather difficult algebraic-geometric questions (in particular, the degree of a projective variety.).

2. The Class of Small Unitary Representations

2.1. **Reductive Lie groups associated with simple Jordan algebras.** Let N be a real simple Jordan algebra with unit e and norm ϕ. Let L be the subgroup of $GL(N)$ that preserves ϕ up to a scalar multiple. Let P be the semidirect product LN, and G the group of rational transformations generated by P and $\iota : x \to -x^{-1}$. Then G is a semisimple Lie group, and $P = LN$ is the Levi decomposition of a parabolic subgroups such that

(i) N is abelian
(ii) $\overline{P}$ is conjugate to P (where $\overline{P}$ is the parabolic of G opposite to P).

Conversely, any parabolic subgroup $P = LN$ of a simple Lie group G satisfying (i) and (ii) arises from a Jordan algebra structure on N. A classification of the such groups will be given below.

2.2. **Restricted roots and multiplicities.** Henceforth, G will be a simple Lie group associated to a real simple Jordan algebra N, with $P = LN$ the corresponding parabolic subgroup of G. Let K be a maximal compact subgroup of G, and let $M = K \cap L$. Let $\mathfrak{g} = Lie\,(G)$, $\mathfrak{k} = Lie\,(K)$, $\mathfrak{n} = Lie\,(N)$, $\mathfrak{m} = Lie\,(M)$, etc. Let $\mathfrak{t}_1$ be a maximal toral subalgebra in the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$. It turns out that there are only three possibilities for the restricted root system Σ of $\mathfrak{t}_1$ in $\mathfrak{k}$; namely, Σ is either of type A_{n-1}, D_n, or C_n. Moreover, there is a fairly uniform prescription for writing down the restricted root systems $\Delta\,(\mathfrak{k};\mathfrak{t}_1)$ and $\Delta\,(\mathfrak{g};\mathfrak{t}_1)$. This goes as follows.

There is a orthonormal basis $\{\gamma_1, \dots, \gamma_n\}$ of $\mathfrak{t}_1^*$ such that the simple roots of $\mathfrak{t}_1$ in $\mathfrak{k}$ are

$$\begin{array}{ll} \left\{\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_{i+1} \mid i = 1, \dots, n-1\right\} & \text{if } \Sigma = A_{n-1} \\ \left\{\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_{i+1} \mid i = 1, \dots, n-1\right\} \cup \{\gamma_n\} & \text{if } \Sigma = C_n \\ \left\{\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_{i+1} \mid i = 1, \dots, n-1\right\} \cup \left\{\frac{1}{2}\gamma_{n-1} + \frac{1}{2}\gamma_n\right\} & \text{if } \Sigma = D_n \end{array} \tag{2.1}$$

and that the roots of $\mathfrak{t}_1$ in $\mathfrak{g}$ are $\left\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j, \pm\gamma_i\right\}$ in **all** cases. Moreover, it turns out, that for a given G, both the short roots $\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j$, and the long roots γ_i have fixed multiplicities. Accordingly, we define integers d and e by

$$\begin{array}{lcl} d & = & \text{common multiplicity of short roots } \pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j \text{ in } \mathfrak{k} \\ e & = & \text{common multiplicity of long roots } \pm\gamma_i \text{ in } \mathfrak{k} \end{array}$$

and the multiplicities of the (restricted) short and long roots of $\mathfrak{t}_1$ in $\mathfrak{g}$ are then, respectively, $2d$ and $e+1$. Let n be the dimension of $\mathfrak{t}_1$ (which coincides with the real rank of $\mathfrak{g}$); it so happens that there is also a convenient formula for $m = \dim\,(\mathfrak{n})$.

$$m = n\,(d\,(n-1) + e + 1)$$

Lemma 2.1. *Let $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_0$ be an extension of $\mathfrak{t}_1$ to a CSA for $\mathfrak{k}$ (with $\mathfrak{t}_0 \subset \mathfrak{m}$). For $\underline{\alpha} \in \mathfrak{t}^* \simeq \mathfrak{t}_1^* \oplus \mathfrak{t}_0^*$, we will write $\underline{\alpha} = (\alpha;\mu)$, with $\alpha \in \mathfrak{t}_1^*$, $\mu \in \mathfrak{t}_2^*$. Then the positive system Σ^+ corresponding to the choice (1) of simple roots in Σ can extended to a positive system $\Delta^+ = \Delta^+\,(\mathfrak{k};\mathfrak{t})$ for $\Delta\,(\mathfrak{k};\mathfrak{t})$, in such a way that a root $(\alpha;\mu)$ is positive in Δ^+ if $\alpha \in \Sigma^+$.*

Remark 2.2. Write $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ for the Cartan decomposition of $\mathfrak{g}$. It turns out that the weights $\{(\pm\gamma_1;0), (\pm\gamma_2;0), \dots (\pm\gamma_n;0)\}$ are the extremal weights of representation of K carried by $\mathfrak{p}$. Indeed, for the positive system Δ^+ arising from positive system (1) of Σ, the highest weight of the representation of K on s is $(\gamma_1;0)$ and that the weights

$$\lambda_i = (\gamma_1;0) + (\gamma_2;0) + \cdots + (\gamma_i;0)$$

are dominant integral for Δ^+.

Indeed, let $\mathfrak{t}$ be an arbitrary Cartan subalgebra of $\mathfrak{k}$, and Δ^+ a positive system for $\Delta\,(\mathfrak{k};\mathfrak{t})$. Then set $\underline{\gamma_1}$ equal to the highest root of the representation of K on $\mathfrak{p}$ and let $W \cdot \underline{\gamma_1}$ be the Weyl orbit of $\underline{\gamma_1}$. Then there will be a unique root $\underline{\gamma_2}$ in the $W \cdot \underline{\gamma_1}$ that is orthogonal to $\underline{\gamma_1}$ and is such that $\underline{\gamma_1} + \underline{\gamma_2}$ is dominant integral. And there will be a unique root $\underline{\gamma_3} \in W \cdot \underline{\gamma_1}$, that is perpendicular to both $\underline{\gamma_1}$ and $\underline{\gamma_2}$ and such that $\underline{\gamma_1} + \underline{\gamma_2} + \underline{\gamma_3}$ is dominant integral. Continuing in this manner, we can construct a sequence $\left\{\underline{\gamma_1}, \dots, \underline{\gamma_n}\right\}$ of $\mathfrak{p}$-roots. We can now set $\mathfrak{t}_1 = span\left\{\left[\mathfrak{p}_{\underline{\gamma_i}}, \mathfrak{p}_{-\underline{\gamma_i}}\right] \mid i = 1, \dots, n\right\}$, and we're back where we started from with $\gamma_i = \underline{\gamma_i}\Big|_{\mathfrak{t}_1}$.

Below is a tables of the simple Lie groups corresponding to non-euclidean real simple Jordan algebras, and the corresponding values d and e. (Henceforth, we are going to drop from consideration the cases where $\Sigma = A_{n-1}$, which correspond to the cases where N is a formally real Jordan algebra.)

G	K	Σ_n	d	e	$m = \dim \mathfrak{n}$
SL_{2n}	SO_{2n}	D_n	1	0	n^2
$SO^o_{2n,2n}$	$SO_{2n} \times SO_{2n}$	D_n	2	0	$2n^2 - n$
$E_7(7)$	SU_8	D_3	3	4	33
$SO_{p,q}$	$SO_p \times SO_q$	D_2	2	$\frac{p+q-4}{2}$	$2 + p + q$
$Sp(n, \mathbb{C})$	Sp_n	C_n	1	1	$n^2 + n$
$SL(2n, \mathbb{C})$	SU_{2n}	C_n	2	1	$2n^2$
$SO(4n, \mathbb{C})$	SO_{4n}	C_n	4	1	$4n^2 - 2n$
$E_7(\mathbb{C})$	E_7	C_3	3	8	45
$SO(p, \mathbb{C})$	SO_p	C_2	2	$p - 4$	$2p - 2$
$Sp(n, n)$	$Sp_n \times Sp_n$	C_n	2	2	$4n^2$
$SL(2n, \mathbb{H})$	Sp_{2n}	C_n	4	3	$4n^2$
$SO(p, 1)$	$SO(p)$	C_1	0	$p - 1$	p

2.3. **BSZ and Sahi's representations.** Put $r = d(n-1) + e$ and let ν be the positive character for L such that ν^{2r} is the determinant of the adjoint action of L on $\mathfrak{n}$. For $t \in \mathbb{R}$, let $(\Pi_t, I(t))$ denote the (normalized) induced representation $Ind_P^G(\nu^t)$.

Theorem 2.3. *Assume $\Sigma(\mathfrak{k}, \mathfrak{t}_1)$ is not of type D_2. Then $I(t)$ has an unitarizable constituent Π_i of rank $i - 1 < n$, if and only $t = d(n-i) + e + 1$. Moreover, for this value of t, this unitarizable constituent is actually a submodule, and its K-types are*

$$\left\{ \lambda = \sum a_j \gamma_j \mid a_i = a_{i+1} = \cdots = a_n = 0 \right\}$$

and these K-types occur with mulitplicity one.[1]

Notation 2.4. Henceforth, G will consistently denote one of the groups listed in Table 1, n will denote its real rank (which happens to equal to the rank of the corresponding Jordan algebra), and Π_i, $i \in \{1, \ldots, n\}$ will denote the irreducible unitarizable constituent of the corresponding induced representation as prescribed in the preceding theorem.

3. Asymptotics of the K-types of Π_i

Let X be the Harish-Chandra module of an irreducible admissible representation of a reductive Lie group G. A filtration

$$\{0\} \subset X_0 \subset X_1 \subset X_2 \subset \cdots \tag{3.1}$$

of X by subspaces X_i is said to be a *good filtration* ([V]) if

(i) $\dim X_n < \infty$
(ii) each X_n is K-invariant
(iii) $\mathfrak{g} \cdot X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$.

Given a good filtration (3.1) of Harish-Chandra module X, one has a corresponding graded object

$$gr(X) = \bigoplus_{n=0}^{\infty} X_n / X_{n-1}$$

which is a finitely generated, graded $(S(\mathfrak{p}), K)$-module. As such, by a theorem of Hilbert and Serre, there exists a polynomial $\varphi(n)$ such that

$$\varphi_{X(n)} = \sum_{q \geq n} \dim_{\mathbb{C}} (X_q / X_{q-1})$$

[1] Here and henceforth we will use γ_i to denote either a restricted root in Σ or its (trivial) extension to $\Delta(\mathfrak{k}; \mathfrak{t})$.

Write

$$\varphi_X(n) = \frac{B_X}{D!} n^D + \text{ lower order terms} \tag{3.2}$$

Then, the leading power D of φ_X is the *Gelfand-Kirillov* dimension of X and the leading coefficient of φ_X times $D!$ is the *Bernstein degree* of X.

3.1. **Initial Formula for Leading Term of** $p_{\Pi_i}(k)$**.** Let Π_i be the Harish-Chandra module of the irreducible unitarizable constitute of $Ind_P^G(\nu^t)$, occuring for $t = d(n-i) + e + 1$. Its K-types[2] are

$$\Lambda = \left\{ \lambda = \sum_{j=1}^{i} a_j \gamma_j \mid a_i \in \mathbb{N} \quad ; \quad a_1 \geq a_2 \geq \cdots \geq a_i \geq 0 \right\}$$

if $i < n$, $i = n$ and $\Sigma = C_n$, or

$$\Lambda = \left\{ \lambda = \sum_{j=1}^{i} a_j \gamma_j \mid a_i \in \mathbb{N} \quad ; \quad a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n| \geq 0 \right\}$$

if $i = n$ and $\Sigma = D_n$.

We filter Π_i as follows

$$V_n = \{K\text{-types } \lambda = a_1\gamma_1 + \cdots + a_i\gamma_i \text{ of } \Pi_i \text{ such that } a_1 \leq n\}$$

This is a good filtration of V.(We note that each V_n is finite-dimensional and obviously stable under K, and because the highest weight of $\mathfrak{s}$ is γ_1, the only K-types that can occur in $\mathfrak{g}{\cdot}V_n$ are K-types $\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$ with $a_1 \leq n+1$.)

We will now compute the leading term of the Hilbert polynomial for Π_i :

$$p_{\Pi_i}(k) = \dim(V_k) = \sum_{a_1=0}^{k} \sum_{a_2=0}^{a_1} \cdots \sum_{a_i=0}^{a_{i-1}} \dim F_{a_1\gamma_1 + \cdots a_i\gamma_i}$$

Here F_λ denotes the irreducible finite dimensional representation of K with highest weight λ. If $\Sigma = D_n$ and $i = n$ the summation on the far right should actually run from $a_n = -a_{n-1}$ to a_{n-1}. Shortly, we'll introduce a special factor to keep track of this special case.

According to the Weyl dimension formula, the dimension of F_λ is

$$\dim F_\lambda = \prod_{\alpha \in \Delta_K^+} \frac{\langle \lambda + \rho_K, \alpha \rangle}{\langle \rho_K, \alpha \rangle}$$

In order to use this formula to compute the dimensions of the K-types $\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$ appearing in Π_i, we shall split the set Δ_K^+ into three disjoint sets:

$$\Delta_K^+ = \Delta_{\Sigma,i}^+ \cup \Delta_{\Sigma,i,c}^+ \cup \Delta_M^+$$

where

$$\begin{aligned}
\Delta_{\Sigma,i}^+ &= \left\{ \alpha \in \Delta_\Sigma^+ \mid \langle \alpha, \gamma_j \rangle \neq 0 \text{ for some } j \in \{1, 2, \ldots, i\} \right\} \\
\Delta_{\Sigma,i,c}^+ &= \left\{ \alpha \in \Delta_\Sigma^+ \mid \langle \alpha, \gamma_j \rangle = 0 \text{ for all } j \in \{1, 2, \ldots, i\} \right\} = \Delta_\Sigma^+ - \Delta_{\Sigma,i}^+ \\
\Delta_M^+ &= \left\{ \alpha \in \Delta_K^+ \mid \langle \alpha, \gamma_i \rangle = 0 \text{ for all } j \in \{1, \ldots, n\} \right\}
\end{aligned}$$

[2] We specify K-types by their highest weight vectors.

We can now, accordingly, break up the product over the elements of Δ_K^+ in the Weyl dimension formula

$$\begin{aligned}
\dim F_{a_1\gamma_1+\cdots+a_i\gamma} &= \left(\prod_{\alpha\in\Delta_{\Sigma,i}^+} \frac{\langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle}{\langle\rho_K,\alpha\rangle}\right)\left(\prod_{\alpha\in\Delta_M^+\cup\Delta_{\Sigma,i,c}^+} \frac{\langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle}{\langle\rho_K,\alpha\rangle}\right) \\
&= \left(\prod_{\alpha\in\Delta_{\Sigma,i}^+} \frac{\langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle}{\langle\rho_K,\alpha\rangle}\right)\left(\prod_{\alpha\in\Delta_M^+\cup\Delta_{\Sigma,i,c}^+} \frac{\langle\rho_K,\alpha\rangle}{\langle\rho_K,\alpha\rangle}\right) \\
&= \prod_{\alpha\in\Delta_{\Sigma,i}^+} \frac{\langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle}{\langle\rho_K,\alpha\rangle} \\
&= \frac{\left(\prod_{\alpha\in\Delta_{\Sigma,i}^+}\langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle\right)}{\left(\prod_{\alpha\in\Delta_{\Sigma,i}^+}\langle\rho_K,\alpha\rangle\right)}
\end{aligned}$$

Thus, the Hilbert polynomial for Π_i will be

$$\begin{aligned}
p_{\Pi_i}(k) &= \sum_{a_1=0}^{k}\cdots\sum_{a_i}^{a_{i-1}} \dim F_{a_1\gamma_1+\cdots+a_i\gamma_i} \\
&= \sum_{a_1=0}^{k}\cdots\sum_{a_i}^{a_{i-1}} \prod_{\alpha\in\Delta_{\Sigma_i}} \left(\frac{\langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle}{\langle\rho_K,\alpha\rangle}\right) \\
&= \sum_{a_1=0}^{k}\cdots\sum_{a_i}^{a_{i-1}} \prod_{\alpha\in\Delta_{\Sigma_i}} \left(\frac{\langle a_1\gamma_1+\cdots+a_i\gamma_i,\alpha\rangle}{\langle\rho_K,\alpha\rangle}\right) + \text{ lower order terms} \\
&= \frac{1}{\left(\prod_{\alpha\in\Delta_{\Sigma_i}}\langle\rho_K,\alpha\rangle\right)} \sum_{a_1=0}^{k}\cdots\sum_{a_i}^{a_{i-1}} \left(\prod_{\alpha\in\Delta_{\Sigma_i}} \langle a_1\gamma_1+\cdots+a_i\gamma_i,\alpha\rangle\right) + \text{ lower order terms}
\end{aligned}$$

Now if $\Sigma = D_n$ and $i = n$ we actually have

$$\begin{aligned}
p_{\Pi_n}(k) &= \sum_{a_1=0}^{k}\cdots\sum_{a_n=-a_{n-1}}^{a_{n-1}} \dim F_{a_1\gamma_1+\cdots+a_n\gamma_n} \\
&= \sum_{a_1=0}^{k}\cdots\sum_{a_{n-1}=0}^{a_{n-2}} \left(\sum_{a_n=0}^{a_{n-1}} \dim F_{a_1\gamma_1+\cdots+a_m\gamma_n} + \sum_{a_n=-a_{n-1}}^{-1} \dim F_{a_1\gamma_1+\cdots+a_m\gamma_n}\right) \\
&= \sum_{a_1=0}^{k}\cdots\sum_{a_{n-1}=0}^{a_{n-2}} \left(\sum_{a_n=0}^{a_{n-1}} \dim F_{a_1\gamma_1+\cdots+a_m\gamma_n} + \sum_{a_n=1}^{a_{n-1}} \dim F_{a_1\gamma_1+\cdots+a_m\gamma_n}\right) \\
&= \sum_{a_1=0}^{k}\cdots\sum_{a_{n-1}=0}^{a_{n-2}} \left(\sum_{a_n=0}^{a_{n-1}} 2\dim F_{a_1\gamma_1+\cdots+a_m\gamma_n} - \dim F_{a_1\gamma_1+\cdots+a_{n-1}\gamma_n+0}\right) \\
&= \sum_{a_1=0}^{k}\cdots\sum_{a_{n-1}=0}^{a_{n-2}} \left(\sum_{a_n=0}^{a_{n-1}} 2\dim F_{a_1\gamma_1+\cdots+a_m\gamma_n}\right) + \text{ lower order terms.} \\
&= \frac{2}{\left(\prod_{\alpha\in\Delta_{\Sigma_i}}\langle\rho_K,\alpha\rangle\right)} \sum_{a_1=0}^{k}\cdots\sum_{a_n}^{a_{n-1}} \left(\prod_{\alpha\in\Delta_{\Sigma_i}} \langle a_1\gamma_1+\cdots+a_n\gamma_n,\alpha\rangle\right) + \text{ lower order terms}
\end{aligned}$$

Introducing a factor

$$\varepsilon_{G,i} = \begin{cases} 2 & \text{if } i = n \text{ and } \Sigma \sim D_n \\ 1 & \text{otherwise} \end{cases}$$

to accommodate the special case when $\Sigma = D_n$ and $i = n$, we thus have the following formula for the Hilbert polynomial of Π_i

$$p_{\Pi_i}(k) = \frac{\varepsilon_{G,i}}{\left(\prod_{\alpha\in\Delta_{\Sigma_i}} \langle\rho_K,\alpha\rangle\right)} \sum_{a_1=0}^{k}\cdots\sum_{a_i}^{a_{i-1}}\left(\prod_{\alpha\in\Delta_{\Sigma_i}} \langle a_1\gamma_1+\cdots+a_i\gamma_i,\alpha\rangle\right) + \text{ lower order terms}$$

Before preceding further we set

$$\begin{aligned} D_{G,i} &= \prod_{\alpha\in\Delta_{\Sigma_i}} \langle\rho_K,\alpha\rangle \\ N_{G,i}(k) &= \sum_{a_1=0}^{k}\cdots\sum_{a_i}^{a_{i-1}}\left(\prod_{\alpha\in\Delta_{\Sigma_i}} \langle a_1\gamma_1+\cdots+a_i\gamma_i,\alpha\rangle\right) \end{aligned}$$

so that

$$p_{\Pi_i}(k) = \frac{\varepsilon_{G,i}}{D_{G,i}} N_{G,i}(k) + \text{ lower order terms}$$

Clearly, the Gelfand-Kirillov dimension is tied up in the asymptotic behavior of $N_{G,i}(k)$ as $k\to\infty$, while all three factors contribute to the Bernstein degree of Π_i. Indeed, if

$$N_{G,i}(k) = ck^d + \text{lower order terms}$$

then

$$GK\dim(\Pi_i) = 2d$$

and so the Bernstein degree of Π_i will be

$$B_{\Pi_i} = \frac{\varepsilon_{G,i} c d!}{D_{G,i}}$$

3.2. **Computation of $N_{G,i}$ and the Gelfand-Kirillov dimension of Π_i.** In this subsection we shall compute the constants c and d such that

$$N_{G,i}(k) \equiv \sum_{a_1=0}^{k}\cdots\sum_{a_i}^{a_{i-1}}\left(\prod_{\alpha\in\Delta_{\Sigma_i}} \langle a_1\gamma_1+\cdots+a_i\gamma_i,\alpha\rangle\right) = ck^d + \text{lower order terms}$$

Recall that we have the following description of the restricted system $\Sigma(\mathfrak{k};\mathfrak{t})$ in terms of the γ_j :

- Short roots of the form $\frac{\gamma_i+\gamma_j}{2}$, which occur with a common multiplicity d.
- Long roots of the form γ_i, which occur with a common multiplicity e.

Writing

$$\mathfrak{t}^* = \mathfrak{t}_1^* \oplus \mathfrak{t}_0^*$$

and according split the roots in $\underline{\alpha} \in \Delta_K = \Delta(\mathfrak{k};\mathfrak{t})$ into two components

$$\underline{\alpha} = (\alpha;\mu) \qquad , \qquad \alpha\in\mathfrak{t}_1^* \quad , \quad \mu\in\mathfrak{t}_0^*$$

and extend the positive system Σ^+ to a positive system for $\Delta(\mathfrak{k};\mathfrak{h})$.

Formula 3.1. *If*

$$\prod_{\alpha\in\Delta_{\Sigma_i}^+} (\langle a_1\gamma_1+\cdots+a_i\gamma_i,\alpha\rangle) = \left(\frac{1}{2}\right)^{di(2n-i-1)} N_{a_1,\ldots,a_i}$$

where

$$N_{a_1,\ldots,a_i} = \left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{i}\left((a_j)^2-(a_{j'})^2\right)^d\right)\left(\prod_{j=1}^{i}(a_j)^{2d(n-i)+e}\right)$$

Proof. Let

$$\begin{aligned}
\Sigma_{i,1}^{+} &= \left\{\frac{\gamma_j+\gamma_{j'}}{2} \mid 1\le j<j'\le i\right\}\\
\Sigma_{i,2}^{+} &= \left\{\frac{\gamma_j+\gamma_{j'}}{2} \mid 1\le j\le i<j'\le n\right\}\\
\Sigma_{i,3}^{+} &= \left\{\frac{\gamma_j-\gamma_{j'}}{2} \mid 1\le j<j'\le i\right\}\\
\Sigma_{i,4}^{+} &= \left\{\frac{\gamma_j-\gamma_{j'}}{2} \mid 1\le j\le i<j'\le n\right\}\\
\Sigma_{i,5}^{+} &= \left\{\gamma_j \mid 1\le j<i\right\}
\end{aligned}$$

so that

$$\Sigma_i^{+} = \bigcup_{k=1}^{5} \Sigma_{i,k}^{+}$$

Then

$$\begin{aligned}
&\prod_{\alpha\in\Delta_{\Sigma_i}^{+}} \langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle\\
=\;& \left(\prod_{\alpha\in\Sigma_{i,1}^{+}} \langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle\right)^{d}\cdot\left(\prod_{\alpha\in\Sigma_{i,2}^{+}} \langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle\right)^{d}\\
&\cdot\left(\prod_{\alpha\in\Sigma_{i,3}^{+}} \langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle\right)^{d}\cdot\left(\prod_{\alpha\in\Sigma_{i,4}^{+}} \langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle\right)^{d}\\
&\cdot\left(\prod_{\alpha\in\Sigma_{i,5}^{+}} \langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\alpha\rangle\right)^{e}\\
=\;& \left(\prod_{1\le j<j'\le i}\left\langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\frac{\gamma_j-\gamma_{j'}}{2}\right\rangle\left\langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\frac{\gamma_j-\gamma_{j'}}{2}\right\rangle\right)^{d}\\
&\cdot\left(\prod_{1\le j\le i}\prod_{i<j'\le n}\left\langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\frac{\gamma_j+\gamma_{j'}}{2}\right\rangle\left\langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\frac{\gamma_j-\gamma_{j'}}{2}\right\rangle\right)^{d}\\
&\cdot\left(\prod_{j=1}^{n}\langle a_1\gamma_1+\cdots+a_i\gamma_i+\rho_K,\gamma_j\rangle\right)^{e}\\
=\;& \left(\prod_{1\le j<j'\le i}\frac{1}{4}\left(a_j^2-a_{j'}^2\right)\right)^{d}\left(\prod_{1\le j\le i}\frac{1}{4}a_j^{(n-i)2}\right)^{d}\left(\prod_{1\le j\le i}a_j\right)^{e}\\
=\;& \left(\frac{1}{2}\right)^{di(2n-i-1)}\left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{i}\left((a_j)^2-(a_{j'})^2\right)^{d}\right)\left(\prod_{j=1}^{i}(a_j)^{2d(n-i)+e}\right)\\
=\;& \left(\frac{1}{2}\right)^{di(2n-i-1)} N_{a_1,\ldots,a_i}
\end{aligned}$$

∎

We now have

$$N_{G,i}(k) = \left(\frac{1}{2}\right)^{di(2n-i-1)} \sum_{a_1=0}^{k} \cdots \sum_{a_i}^{a_{i-1}} N_{a_1 \dots a_i}$$

but our task remains to compute the constants $C_{G,i}$ and d such that

$$N_{G,i}(k) = ck^d + \text{lower order terms}$$

The next step will be to establish an integral formula for the constant c. Let

$$I(k) \equiv \int_{a_1=0}^{k} da_1 \int_{a_2=0}^{a_1} da_2 \cdots \int_{a_i=0}^{a_{i-1}} N_{a_1,\dots,a_i} da_i$$

Using a Riemann sum argument we can bound the multiple summation between two integrals:

$$I(k) \leq \sum_{a_1=0}^{k} \sum_{a_2=0}^{a_1} \cdots \sum_{a_i=0}^{a_i-1} N_{a_1,\dots,a_i} \leq I(k+1)$$

for all k. Now observe that the integrand $N_{a_1,\dots,a_i}$ is a homogeneous polynomial of total degree $2ndi - di^2 - id + ie$. It is easy to see that a each stage of evaluation the integrand increases its total degree but remains homogeneous. Thus, upon evaluation $I(k)$ will result in a monomial of the form $ck^{2ndi-di^2-id+ie+i}$ and $I(k+1)$ will evaluate to $c(k+1)^{2ndi-di^2-id+ie+i}$, with the same constant c. We thus have

$$ck^{2ndi-di^2-id+ie} \leq \sum_{a_1=0}^{k} \sum_{a_2=0}^{a_1} \cdots \sum_{a_i=0}^{a_i-1} N_{a_1,\dots,a_i} \leq c(k+1)^{2ndi-di^2-id+ie} \qquad , \qquad \text{for all } k > 1$$

Dividing everything by $k^{2ndi-di^2-id+ie}$, we have

$$c \leq \frac{1}{k^{2ndi-di^2-id+ie}} \sum_{a_1=0}^{k} \sum_{a_2=0}^{a_1} \cdots \sum_{a_i=0}^{a_i-1} N_{a_1,\dots,a_i} \leq c + \text{terms of order } (1/k)$$

Now letting $k \to \infty$, we can conclude for sufficiently large k we must have

$$\sum_{a_1=0}^{k} \sum_{a_2=0}^{a_1} \cdots \sum_{a_i=0}^{a_i-1} N_{a_1,\dots,a_i} \sim ck^{2ndi-di^2-id+ie} + \text{ lower order terms}$$

In fact, since $I(k)$ is a monomial in k, the constant c can be obtained by simply evaluating $I(k)$ at $k=1$.

$$\begin{aligned} c &= I(1) = \int_0^1 da_1 \int_0^{a_1} da_2 \cdots \int_0^{a_{i-1}} N_{a_1,\dots,a_i} da_i \\ &= \left(\frac{1}{2}\right)^{di(2n-i-1)} \int_0^1 da_1 \int_0^{a_1} da_2 \cdots \int_0^{a_{i-1}} \left(\prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} \left((a_j)^2 - (a_{j'})^2 \right)^d \right) \left(\prod_{j=1}^{i} (a_j)^{2d(n-i)+e} \right) da_i \end{aligned}$$

Note that in the preceding calculation we have shown that the dimension of the K-types of Π_i grow like $k^{di(2n-i)+i(e-d)}$. Thus,

Proposition 3.2. *The Gelfand-Kirillov dimension of* Π_i *is*

$$D = 2di(2n-i) + 2i(e-d)$$

We also the following formula for the leading term of $N_{G,i}(k)$

$$N_{G,i}(k) = C_{G,i} k^{di(2n-i)+i(e-d)} + \text{lower order terms}$$

with the constant $C_{G,i}$ determined by the integral formula

$$C_{G,i} \equiv \frac{1}{2^{di(2n-i-1)}} \int_0^1 da_1 \int_0^{a_1} da_2 \cdots \int_0^{a_{i-1}} \left(\prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} \left((a_j)^2 - (a_{j'})^2 \right)^d \right) \left(\prod_{j=1}^{i} (a_j)^{2d(n-i)+e} \right) da_i$$

We'll now procede with the evaluation of the integral in the formula for $C_{G,i}$. Set

$$I_o(i,d,p) \equiv \int_0^1 da_1 \int_0^{a_1} da_2 \cdots \int_0^{a_{i-1}} da_i \left(\prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} \left((a_j)^2 - (a_{j'})^2 \right)^d \right) \left(\prod_{j=1}^{i} (a_j)^p \right)$$

After a change of variables, $x_j = (a_j)^2$, $j = 1, \ldots, i$ we have

$$I(i,d,p) = \frac{1}{2^i} \int_{a_1=0}^1 dx_1 \cdots \int_0^{x_{i-1}} dx_i \left(\prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} (x_j - x_{j'})^{\frac{d}{2}} \right) \left(\prod_{j=1}^{i} (x_j)^{\frac{p-1}{2}} \right)$$

To evaluate this integral we can employ Selberg's integral formula (as stated by I.G. Macdonald, *Some conjectures for root systems*, SIAM J. Math. Anal., **13** (1982), 988-1007).

Lemma 3.3 (Selberg, Macdonald). *Let*

$$J_i(a,b;c) = \int_{C^n} \prod_{j=1}^{i} \left(x_j^a (1-x_j)^b \right) |D(x)|^{2c} dx_1 \cdots dx_i$$

where $C^i = \left\{ (x_1, \ldots, x_i) \in \mathbb{R}^i \mid 0 \leq x_j \leq 1 \quad , \quad j = 1, \ldots, i \right\}$,

$$D = \prod_{1 \leq j < j' \leq i} (x_j - x_{j'}) \quad ,$$

and a, b, c *are complex numbers satisfying*

$$\begin{aligned} \mathrm{Re}(a) &> -1 \\ \mathrm{Re}(b) &> -1 \\ \mathrm{Re}(c) &> -\min\left(\frac{1}{n}, \frac{\mathrm{Re}(a+1)}{n-1}, \frac{\mathrm{Re}(b+1)}{n-1} \right) \end{aligned}$$

Then

$$J_i(a,b;c) = \prod_{j=1}^{i} \frac{\Gamma(jc+1)\Gamma(a+(k-1)c+1)\Gamma(b+(j-1)c+1)}{\Gamma(c+1)\Gamma(a+b+2+(i+j-2)c)}$$

We can readily massage the Selberg integral formula to handle our particular integral. First of all we note that the integrand of $(*)$ coincides with that of the Selberg integral when we take $a = (p-1)/2$, $b = 0$, and $c = d/2$ and restrict the domain of integration to $\left\{ x \in \mathbb{R}^i \mid 0 \leq x_i \leq x_{i-1} \leq \cdots \leq x_2 \leq x_1 \leq 1 \right\}$. We note also that our parameters p and d will always be non-negative real numbers.

Next we partition the domain of integration for Selberg's integral, the unit cube in $\mathbb{R}^i$, into a set of $n!$ disjoint wedge-like regions:

$$\begin{aligned} D_\sigma &= \left\{ x \in \mathbb{R}^i \mid 0 \leq x_{\sigma(i)} \leq x_{\sigma(i-1)} \leq \cdots \leq x_{\sigma(2)} \leq x_{\sigma(1)} \leq 1 \right\} \quad , \quad \sigma \in S_i \\ C_n &= \bigcup_{\sigma \in S_i} D_\sigma \end{aligned}$$

Note how we have used the permutation group S_i to parameterize sets D_σ. We can thus write

$$J_i\left(p, 0; \frac{d}{2}\right) = \sum_{\sigma \in S_i} \int_{D_\sigma} \left(\prod_{j=1}^{i} (x_j)^p \right) |D(x)|^d dx_1 \cdots dx_i$$

But, since the integrand is manifestly invariant with respect to interchanges of variables, we can, by a change of variables, write each of the integrals in the sum as an integral over

$$D_1 = \left\{ x \in \mathbb{R}^i \mid 0 \leq x_i \leq x_{i-1} \leq \cdots \leq x_2 \leq x_1 \leq 1 \right\}$$

We thus obtain

$$J_i\left(p,0;\frac{d}{2}\right)=\sum_{\sigma\in S_i}\int_{D_1}\left(\prod_{j=1}^{i}\left(x_{\sigma(j)}\right)^p\right)\left|D\left(\sigma\left(x\right)\right)\right|^d dx_{\sigma(1)}\cdots dx_{\sigma(i)}$$

But now we note that the integrand is invariant with respect to permutations of variables, and so

$$\begin{aligned}J_i\left(p,0;\frac{d}{2}\right) &= \sum_{\sigma\in S_i}\int_{D_1}\left(\prod_{j=1}^{i}\left(x_{\sigma(j)}\right)^p\right)\left|D\left(\sigma\left(x\right)\right)\right|^d dx_{\sigma(1)}\cdots dx_{\sigma(i)}\\ &= \sum_{\sigma\in S_i}\int_0^1 dx_1\cdots\int_0^{x_{i-1}}dx_i\left(\prod_{j=1}^{i}\left(x_j\right)^p\right)\left|D\left(x\right)\right|^d\\ &= 2^i i!I\left(i,d,p\right)\end{aligned}$$

And so we can conclude that

$$I\left(i,d,p\right)=\frac{1}{2^i i!}J_i\left(p,0;\frac{d}{2}\right)$$

Putting this all together we have,

Proposition 3.4. (i) *The Gelfand-Kirillov dimension D of a representation Π_i is given by*

$$GK\dim\left(\Pi_i\right)=2di\left(2n-1\right)+2i(e-d+1)$$

(ii) *The leading coefficient of $N_{G,i}\left(k\right)$ is*

$$\begin{aligned}C_{G,i} &\equiv \frac{1}{i!2^{i(2dn-d(i+1)+1)}}J_i\left(\frac{2d\left(n-i\right)+e-1}{2},0;\frac{d}{2}\right)\\ &= \frac{1}{i!2^{di(2n-i-1)+i}}\prod_{j=1}^{i}\frac{\Gamma\left(\frac{jd}{2}+1\right)\Gamma\left(d\left(n-i\right)+\frac{e+1+(j-1)d}{2}\right)\Gamma\left(\frac{d(j-1)}{2}+1\right)}{\Gamma\left(\frac{d}{2}+1\right)\Gamma\left(d\left(n-i\right)+\frac{e+3+(i+j-2)d}{2}\right)}\end{aligned}$$

3.3. **Evaluating $D_{G,i}$.** Recall

$$D_{G,i}\equiv\prod_{\alpha\in\Delta^+_{\Sigma,i}}\left\langle\rho_K,\alpha\right\rangle$$

where

$$\Delta^+_{\Sigma,i}=\left\{\alpha\in\Delta^+\left(\mathfrak{k};\mathfrak{t}_1+\mathfrak{t}_0\right)\mid\left\langle\alpha,\gamma_j\right\rangle\neq 0\text{ for some }j\in\{1,\ldots,i\}\right\}$$

Once one has expressions for the γ_i in terms of the basis vectors $\{e_i\}$ of the Euclidean space in which the roots of $\mathfrak{k}$ are usually expressed, the $\Delta^+_{\Sigma,i}$ are evaluated fairly easily.

Below we present a summary of the results of these calculations; we'll confine the explicit calculations of the denominators to an appendix.

4. Summary: Gelfand-Kirillov dimensions and Bernstein degrees

Theorem 4.1. *Let G be one of the Lie groups listed in Table 1. Let n denote the real rank of G, and let d and e denote, respectively, the common multiplicity of the short and long roots in the restricted root system $\Delta\left(\mathfrak{k};\mathfrak{t}_1\right)$. Let Π_i, $i=1,\ldots,n$ be the $(\mathfrak{g},K)$-module of the unitarizable constitutent of one of Sahi's representations. Then*

- The Gelfand-Kirillov of Π_i is

$$GK\dim\left(\Pi_i\right)=2di\left(2n-1\right)+2i(e-d+1)$$

- The Bernstein degree of Π_i is

$$B_{\Pi_i} = \frac{\varepsilon_\Sigma(i)\ \ (di(2n-1)+i(e-d+1))!C_{G,i}}{\left(\prod_{\alpha\in\Delta_{\Sigma_i}}\langle\rho_K,\alpha\rangle\right)}$$

where

$$\varepsilon_\Sigma(i) = \begin{cases} 2 & \text{if } i=n \text{ and } \Sigma\sim D_n \\ 1 & \text{otherwise}\end{cases}$$

$$C_{G,i} = \frac{1}{i!2^{di(2n-i-1)+i}}\prod_{j=1}^{i}\frac{\Gamma\left(\frac{jd}{2}+1\right)\Gamma\left(d(n-i)+\frac{e+1+(j-1)d}{2}\right)\Gamma\left(\frac{d(j-1)}{2}+1\right)}{\Gamma\left(\frac{d}{2}+1\right)\Gamma\left(d(n-i)+\frac{e+3+(i+j-2)d}{2}\right)}$$

and the remaining data is contained in the following table:

G	K	Σ_n	d	e	$D_{G,i}=\prod_{\alpha\in\Delta^+_{\Sigma,i}}\langle\rho_K,\alpha\rangle$
SL_{2n}	SO_{2n}	D_n	1	0	$\frac{(n-1)!}{(n-i-1)!}\prod_{j=1}^{i}(2n-2j+1)!$
$SO^o_{2n,2n}$	$SO_{2n}\times SO_{2n}$	D_n	2	0	$\left(\frac{(n-1)!}{(n-i-1)!}\prod_{j=1}^{i}(2n-2j+1)!\right)^2$
$E_7(7)$	SU_8	D_3	3	4	$2^{12}3^5 5^3 7$ if $i=1$; $2^{16}3^7 5^3 7$ if $i=2,3$
$SO_{p,q}$	$SO_p\times SO_q$	D_2	2	$\frac{p+q-4}{2}$	$\frac{1}{4}(p-2)!(q-2)!$
$Sp(n,\mathbb{C})$	Sp_n	C_n	1	1	$2^i\left(\prod_{j=1}^{i}(2n-2j+3)!\right)$
$SL(2n,\mathbb{C})$	SU_{2n}	C_n	2	1	$\left(\frac{1}{2}\right)^{i(4n-2i-1)}\left(\prod_{j=1}^{2n-1}(2n-j)!\right)\left(\prod_{j=1}^{2n-2i}\frac{1}{(2n-2i-j)!}\right)$
$SO(4n,\mathbb{C})$	SO_{4n}	C_n	4	1	$\prod_{j=1}^{2i}\left(\frac{(4n-2j+1)!}{(2n-j-1)!}(2n-j)!\right)$
$E_7(\mathbb{C})$	E_7	C_3	3	8	$2^{29}3^{13}5^6 7^4 11^3 13^2 17$ if $i=1$; $2^{42}3^{19}5^9 7^6 11^3 13^2 17$ if $i=2,3$
$SO(p,\mathbb{C})$	SO_p	C_2	2	$p-4$	$\prod_{j=1}^{2i}\frac{1}{2}(p-2j)(p-2j+1)!$
$Sp(n,n)$	$Sp_n\times Sp_n$	C_n	2	2	$2^{2i}\left(\prod_{j=1}^{i}(2n-2j+3)!\right)^2$
$SL(2n,\mathbb{H})$	Sp_{2n}	C_n	4	3	$2^{2i}\frac{(2n)!}{(2n-2i)!}\left(\prod_{j=1}^{2i}\left(\frac{(4n-2j+3)!}{(2n-j+1)!}(n-j)!\right)\right)$
$SO(p,1)$	$SO(p)$	C_1	0	$p-1$	$\frac{1}{2}(p-2)!$

Appendix A. Explicit Calculations of the Denominators $D_{G,i}$

A.1. **Useful Formulas.** Here's a table from Bourbaki

A$_n$ $\Delta^+=\{e_i\pm e_j \mid 1\le i<j\le n+1\}$
$\rho=\frac{1}{2}(ne_1+(n-2)e_2+\cdots+(2-n)e_n-ne_{n+1})$
B$_n$ $\Delta^+=\{e_i \mid 1\le i<n\}\cup\{e_i\pm e_j \mid 1\le i<j\le n\}$
$\rho=\frac{1}{2}((2n-1)e_1+(2n-3)e_2+\cdots+3e_{n-1}+e_n)$
C$_n$ $\Delta^+=\{2e_i \mid 1\le i<n\}\cup\{e_i\pm e_j \mid 1\le i<j\le n\}$
$\rho=ne_1+(n-1)e_2+\cdots+2e_{n-1}+e_n$
D$_n$ $\Delta^+=\{e_i\pm e_j \mid 1\le i<j\le n\}$
$\rho=(n-1)e_1+(n-2)e_2+\cdots+e_{n-1}$
E$_7$ $\Delta^+=\{\pm e_i+e_j \mid 1\le i<j\le j\}\cup\{e_8-e_7\}\cup\left\{\frac{1}{2}(e_8-e_7)+\sum_{i=1}^{6}(-1)^{v(i)}e_i \mid \sum_{i=1}^{6}v(i)\right\}$
$\rho=e_2+2e_3+3e_4+4e_5+5e_6-\frac{17}{2}e_7+\frac{17}{2}e_8$

Formula A.1.

(A) $$\prod_{j=1}^{i-1}\prod_{j'=j+1}^{i}(N-j-j')(j'-j)=\prod_{j=1}^{i-1}\left(\frac{(N-2j+1)!}{(N-i-j-1)!}(i-j)!\right)$$

Proof.

$$\begin{aligned}
LHS &= \prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} (N-j-j')(j'-j) \\
&= \prod_{j=1}^{i-1} (N-2j-1)(1)(N-2j-2)(2)\cdots(N-i-j+1)(i-j-1)(N-i-j)(i-j) \\
&= (N-3)(1)(N-4)(2)\cdots(N-i)(i-2)(N-i-1)(i-1) \\
&\quad \cdot (N-5)(1)(N-6)(2)\cdots(N-i-1)(i-2)(N-i-2)(i-2) \\
&\quad \vdots \\
&\quad (N-2i+3)(1)(N-2i+2)(2) \\
&\quad (N-2i+1)(1) \\
&= \left(\frac{(N-3)!}{(N-i-2)!}(i-1)!\right)\left(\frac{(N-5)!}{(N-i-3)!}(i-2)!\right)\cdots\left(\frac{(N-2i+3)!}{(N-2i+1)!}(2)!\right)\left(\frac{(N-2i+1)!}{(N-2i)!}(1)!\right) \\
&= \prod_{j=1}^{i-1}\left(\frac{(N-2j+1)!}{(N-i-j-1)!}(i-j)!\right) = RHS
\end{aligned}$$

Formula A.2.

(B) $$\prod_{j=1}^{i} \prod_{j'=i+1}^{n} (N-j-j')(j'-j) = \prod_{j=1}^{i}\left(\frac{(N-i-j-1)!}{(N-n-j-1)!}\frac{(n-j)!}{(i-j)!}\right)$$

Proof.

$$\begin{aligned}
LHS &= \prod_{j=1}^{i} \prod_{j'=i+1}^{n} (N-j-j')(j'-j) \\
&= \prod_{j=1}^{i} (N-i-j-1)(i-j+1)(N-i-j-2)(i-j+2)\cdots(N-n-j+1)(n-1-j)(N-n-j)(n-j) \\
&= (N-i-2)(i)(N-i-3)(i+1)\cdots(N-n)(n-2)(N-n-1)(n-1) \\
&\quad \cdot (N-i-3)(i-1)(N-2-i-2)(i+2-2)\cdots(N-n-1)(n-3)(N-n-2)(n-2) \\
&\quad \vdots \\
&\quad \cdot (N-2i-2)(2)(N-2i-3)(3)\cdots(N-n-i+2)(n-i)(N-n-i+1)(n-i+1) \\
&\quad \cdot (N-2i-1)(1)(N-2i-2)(2)\cdots(N-n-i+1)(n-i-1)(N-n-i)(n-i) \\
&= \left(\frac{(N-i-2)!}{(N-n-2)!}\frac{(n-1)!}{(i-1)!}\right)\left(\frac{(N-i-3)!}{(N-n-3)!}\frac{(n-i+1)!}{(i-2)!}\right)\cdots\left(\frac{(2n-2i)!}{(n-i)!}\frac{(n-i+1)!}{1!}\right) \\
&\quad \cdot \left(\frac{(2n-2i-1)!}{(n-i-1)!}\frac{(n-i)!}{0!}\right) \\
&= \prod_{j=1}^{i}\left(\frac{(N-i-j-1)!}{(N-n-j-1)!}\frac{(n-j)!}{(i-j)!}\right) = RHS
\end{aligned}$$

Formula A.3.

(C)

$$\left(\prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} (N-j-j')(j'-j)\right)\left(\prod_{j=1}^{i} \prod_{j'=i+1}^{n} (N-j-j')(j'-j)\right) = \prod_{j=1}^{i}\left(\frac{(N-2j+1)!}{(N-n-j-1)!}(n-j)!\right)$$

Proof. *Using Formulas A and B we have*

$$\begin{aligned}
LHS &= \left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{i}(N-j-j')\,(j'-j)\right)\left(\prod_{j=1}^{i}\prod_{j'=i+1}^{n}(N-j-j')\,(j'-j)\right)\\
&= \left(\prod_{j=1}^{i-1}\left(\frac{(N-2j+1)!}{(N-i-j-1)!}\,(i-j)!\right)\right)\left(\prod_{j=1}^{i}\left(\frac{(N-i-j-1)!}{(N-n-j-1)!}\,\frac{(n-j)!}{(i-j)!}\right)\right)\\
&= \left(\prod_{j=1}^{i-1}\left(\frac{(N-2j+1)!}{(N-i-j-1)!}\,(i-j)!\right)\left(\frac{(N-i-j-1)!}{(N-n-j-1)!}\,\frac{(n-j)!}{(i-j)!}\right)\right)\left(\frac{(N-2i-1)!}{(N-n-i-1)!}\,\frac{(n-i)!}{(0)!}\right)\\
&= \left(\prod_{j=1}^{i-1}\left(\frac{(N-2j+1)!}{(N-n-j-1)!}\,(n-j)!\right)\right)\left(\frac{(N-2i-1)!}{(N-n-i-1)!}\,\frac{(n-i)!}{(0)!}\right)\\
&= \prod_{j=1}^{i}\left(\frac{(N-2j+1)!}{(N-n-j-1)!}\,(n-j)!\right) = RHS
\end{aligned}$$

Formula A.4. *(Another version of the preceding formula)*

$$\text{(D)}\qquad \left(\prod_{j=1}^{i}\prod_{j'=j+1}^{n}(N-j-j')\,(j'-j)\right) = \prod_{j=1}^{i}\left(\frac{(N-2j+1)!}{(N-n-j-1)!}\,(n-j)!\right)$$

Proof. In view of the preceding formula suffices to show that

$$\left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{n}(N-j-j')\,(j'-j)\right) = \left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{i}(N-j-j')\,(j'-j)\right)\left(\prod_{j=1}^{i}\prod_{j'=i+1}^{n}(N-j-j')\,(j'-j)\right)$$

$$\begin{aligned}
RHS &= \left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{i}(N-j-j')\,(j'-j)\right)\left(\prod_{j=1}^{i-1}\prod_{j'=i+1}^{n}(N-j-j')\,(j'-j)\right)\left(\prod_{j'=i+1}^{n}(N-i-j')\,(j'-i)\right)\\
&= \left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{n}(N-j-j')\,(j'-j)\right)\left(\prod_{j'=i+1}^{n}(N-i-j')\,(j'-i)\right)\\
&= \prod_{j=1}^{i}\left(\frac{(N-2j+1)!}{(N-n-j-1)!}\,(n-j)!\right) = LHS
\end{aligned}$$

The Calculations, Case by Case

A.2. $G = SL_{2n}$. $\quad K = SO_{2n} \quad \Sigma = D_n$

$$\begin{aligned}
\gamma_1 &= [2,0,\ldots,0] = 2e_1\\
\gamma_2 &= [-2,2,0,\ldots,0] = 2e_2\\
&\ \ \vdots\\
\gamma_{n-2} &= [0,\ldots,0,-2,2,0,0] = 2e_{n-2}\\
\gamma_{n-1} &= [0,\ldots,0,-2,2,2] = 2e_{n-1}\\
\gamma_n &= [0,\ldots,0,-2,2] = 2e_n
\end{aligned}$$

$$a_1\gamma_1+\cdots+a_n\gamma_n = [2a_1-2a_2, 2a_2-2a_3,\ldots,2a_{n-2}-2a_{n21}, 2a_{n-1}-2a_n, 2a_n+2a_{n-1}]$$

$$\rho_K = (n-1)\,e_1 + (n-2)\,e_2 + \cdots + e_{n-1}$$

So

$$\langle \rho_K, e_j \rangle = n - j$$

In this case we have $\gamma_j = 2e_j$, and so

$$\Delta^+_{\Sigma,i} = \{e_j \pm e_{j'} \mid 1 \leq j < j' \leq i\} \cup \{e_j \pm e_{j'} \mid 1 \leq j \leq i \ , \ i+1 < j' \leq n\}$$

Hence,

$$\begin{aligned}
\prod_{\alpha \in \Delta^+_{\Sigma,i}} \langle \rho_K, \alpha \rangle &= \left(\prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} \langle \rho_K, e_j + e_{j'} \rangle \langle \rho_K, e_j - e_{j'} \rangle \right) \left(\prod_{j=1}^{i} \prod_{j'=i+1}^{n} \langle \rho_K, e_j + e_{j'} \rangle \langle \rho_K, e_j - e_{j'} \rangle \right) \\
&= \left(\prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} (n-j+n-j')(j'-j) \right) \left(\prod_{j=1}^{i} \prod_{j'=i+1}^{n} (n-j+n-j')(j'-j) \right) \\
&= \left(\prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} (2n-j-j')(j'-j) \right) \left(\prod_{j=1}^{i} \prod_{j'=i+1}^{n} (2n-j-j')(j'-j) \right) \\
&= \prod_{j=1}^{i} \left(\frac{(2n-2j+1)!}{(n-j-1)!} (n-j)! \right) \\
&= \prod_{j=1}^{i} ((2n-2j+1)! \, (n-j)) \\
&= \frac{(n-1)!}{(n-i-1)!} \prod_{j=1}^{i} (2n-2j+1)!
\end{aligned}$$

where in the final step we applied Formula C with $N = 2n$.

Thus,

$$\prod_{\alpha \in \Delta^+_{\Sigma,i}} \langle \rho_K, \alpha \rangle = \frac{(n-1)!}{(n-i-1)!} \prod_{j=1}^{i} (2n-2j+1)!$$

A.3. $G = SO^o_{2n,2n}$. $\qquad K = SO_{2n} \times SO_{2n} \qquad \Sigma = D_n$

$$\begin{aligned}
\gamma_1 &= [1,0,\cdots,0;1,0,\ldots,0] = e_1 + \tilde{e}_1 \\
\gamma_2 &= [-1,1,0\ldots,0;-1,1,0,\ldots,0] = e_2 + \tilde{e}_2 \\
\gamma_3 &= [0,-1,1,0,\ldots,0;0,-1,1,0,\ldots,0] = e_3 + \tilde{e}_3 \\
&\vdots \\
\gamma_{n-2} &= [0,\ldots,0,-1,1,0;0,\ldots,0,-1,1,0,0] = e_{n-2} + \tilde{e}_{n-2} \\
\gamma_{n-1} &= [0,\ldots,0,-1,1,1;0,\ldots,0,-1,1,1] = e_{n-1} + \tilde{e}_{n-1} \\
\gamma_n &= [0,\ldots,0,-1,1;0,\ldots,0,-1,1] = e_n + \tilde{e}_n
\end{aligned}$$

$$a_1\gamma_1 + \cdots + a_n\gamma_n = [a_1 - a_2, \ldots, a_{n-1} - a_n, a_{n-1} + a_n; a_1 - a_2, \ldots, a_{n-1} - a_n, a_{n-1} + a_n]$$

$$\rho_K = (n-1)\,e_1 + (n-2)\,e_2 + \cdots + e_{n-1} + (n-1)\,\tilde{e}_1 + (n-2)\,\tilde{e}_2 + \cdots + \tilde{e}_{n-1}$$

$$\Rightarrow \qquad \langle \rho_K, e_j \rangle = \langle \rho_K, \tilde{e}_j \rangle = n - j$$

In this case we have

$$\begin{aligned}
\Delta^+_{\Sigma,i} &= \{e_j \pm e_{j'} \mid 1 \leq j < j' \leq i\} \cup \{e_j \pm e_{j'} \mid 1 \leq j \leq i \ , \ i+1 < j' \leq n\} \\
&\quad \cup \{\tilde{e}_j \pm \tilde{e}_{j'} \mid 1 \leq j < j' \leq i\} \cup \{\tilde{e}_j \pm \tilde{e}_{j'} \mid 1 \leq j \leq i \ , \ i+1 < j' \leq n\}
\end{aligned}$$

$$\begin{aligned}
\prod_{\alpha\in\Delta^+_{\Sigma,i}} \langle \rho_K,\alpha\rangle &= \left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{i} \langle e_j+e_{j'},\rho_K\rangle \langle e_j-e_{j'},\rho_k\rangle\right)\left(\prod_{j=1}^{i}\prod_{j'=i+1}^{n} \langle e_j+e_{j'},\rho_K\rangle \langle e_j-e_{j'},\rho_k\rangle\right)\\
&\cdot\left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{i} \langle \tilde e_j+\tilde e_{j'},\rho_K\rangle \langle \tilde e_j-\tilde e_{j'},\rho_K\rangle\right)\left(\prod_{j=1}^{i}\prod_{j'=j+1}^{n} \langle \tilde e_j+\tilde e_{j'},\rho_K\rangle \langle \tilde e_j-\tilde e_{j'},\rho_K\rangle\right)\\
&= \left(\left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{i}(2n-j-j')(j'-j)\right)\left(\prod_{j=1}^{i}\prod_{j'=i+1}^{n}(2n-j-j')(j'-j)\right)\right)^2\\
&= \left(\prod_{j=1}^{i}\frac{(2n-2j+1)!}{(n-j-1)!}(n-j)!\right)^2\\
&= \left(\frac{(n-1)!}{(n-i-1)!}\prod_{j=1}^{i}(2n-2j+1)!\right)^2
\end{aligned}$$

A.4. $G=E_7(7)$. $K=SU_8$ $\Sigma=D_3$

$$\begin{aligned}
\gamma_1 &= [0,0,0,1,0,0,0]=\frac{1}{2}(e_1+e_2+e_3+e_4-e_5-e_6-e_7-e_8)\\
\gamma_2 &= [0,1,0,-1,0,0,1,0]=\frac{1}{2}(e_1+e_2-e_3-e_4+e_5+e_6-e_7-e_8)\\
\gamma_3 &= [0,1,0,0,0,-1,0]=\frac{1}{2}(e_1+e_2-e_3-e_4-e_5-e_6+e_7+e_8)
\end{aligned}$$

$$a_1\gamma_1+a_2\gamma_2+a_3\gamma_3=[0,a_2+a_3,0,a_1-a_2,0,a_2-a_3,0]$$

$$\rho_K=\frac{1}{2}(7e_1+5e_2++3e_3+e_4-e_5-3e_6-5e_7-7e_8)$$

We carried out the calculation of $\prod_{\alpha\in\Delta^+_{\Sigma,1}}\langle\rho_K,\alpha\rangle$ for $i=1,2,3$ using Maple, and obtained the following results.

$$\begin{aligned}
\prod_{\alpha\in\Delta^+_{\Sigma,1}}\langle\rho_K,\alpha\rangle &= 2^{12}3^5 5^3 7\\
\prod_{\alpha\in\Delta^+_{\Sigma,2}}\langle\rho_K,\alpha\rangle &= 2^{16}3^7 5^3 7\\
\prod_{\alpha\in\Delta^+_{\Sigma,3}}\langle\rho_K,\alpha\rangle &= 2^{16}3^7 5^3 7
\end{aligned}$$

A.5. $G=SO_{p,q}$. $K=SO_p\times SO_q$ $\Sigma=D_2$

$$\begin{aligned}
\gamma_1 &= [1,0,\dots,0;1,0,\dots,0]=e_1+\tilde e_1\\
\gamma_2 &= [1,,0\dots,0;-1,0,\dots,0]=e_1-\tilde e_1
\end{aligned}$$

$$a_1\gamma_1+a_2\gamma_2=[a_1+a_2,0,\cdots,0;a_1-a_2,0,\dots,0]$$

There are four case here, corresponding to p and q being even or odd. However, since

$$\begin{aligned}
\Delta^+_{\Sigma,i} &= \left(\Delta^+_{\Sigma,i}\cap\Delta|_{\mathfrak{so}(p)}\right)\cup\left(\Delta^+_{\Sigma,i}\cap\Delta|_{\mathfrak{so}(q)}\right)\\
&\equiv \left(\Delta^+_{\Sigma,i}\right)_p\cup\left(\Delta^+_{\Sigma,i}\right)_q
\end{aligned}$$

we'll have

$$\prod_{\alpha\in\Delta_{\Sigma,i}^{+}}\langle\rho_K,\alpha\rangle=\left(\prod_{\alpha\in\left(\Delta_{\Sigma,i}^{+}\right)_p}\langle\rho_K,\alpha\rangle\right)\left(\prod_{\alpha\in\left(\Delta_{\Sigma,i}^{+}\right)_q}\langle\rho_K,\alpha\rangle\right)$$

and it will suffice to develop formulas for

$$\prod_{\alpha\in\left(\Delta_{\Sigma,i}^{+}\right)_{2k}}\langle\rho_K,\alpha\rangle$$

and

$$\prod_{\alpha\in\left(\Delta_{\Sigma,i}^{+}\right)_{2k+1}}\langle\rho_K,\alpha\rangle$$

Another convenience for the $SO\,(p,q)$ case is that

$$\Delta_{\Sigma,1}^{+}=\Delta_{\Sigma,2}^{+}$$

A.5.1. *Even Case.* $\left(\Delta_{\Sigma,i}^{+}\right)_{2k}$.

$$\begin{aligned}\rho_K &= (k-1)\,e_1+(k-2)\,e_2+\cdots+e_{k-1}\\ &\Rightarrow \quad \langle\rho_K,e_j\rangle=k-j\end{aligned}$$

$$\left(\Delta_{\Sigma,i}^{+}\right)_{2k}=\{e_1+e_j\mid 2\le j\le k\}\cup\{e_1-e_j\mid 2\le j\le k\}$$

And so

$$\begin{aligned}\prod_{\alpha\in\left(\Delta_{\Sigma,i}^{+}\right)_{2k}}\langle\rho_K,\alpha\rangle &= \prod_{j=2}^{k}\langle\rho_K,e_1+e_j\rangle\langle\rho_K,e_1-e_j\rangle\\ &= \prod_{j=2}^{k}(2k-j-1)\,(j-1)\\ &= (k-1)\,(2k-3)!\\ &= \frac{1}{2}\,(2k-2)!\end{aligned}$$

A.5.2. *Odd Case.* $\left(\Delta_{\Sigma,i}^{+}\right)_{2k+1}$.

$$\begin{aligned}\rho_K &= \frac{1}{2}\,(2k-1)\,e_1+\frac{1}{2}\,(2k-3)\,e_2+\cdots+\frac{1}{2}e_{k-1}\\ &\Rightarrow \quad \langle\rho_K,e_j\rangle=\frac{1}{2}\,(2k-2j+1)\end{aligned}$$

$$\left(\Delta_{\Sigma,i}^{+}\right)_{2k}=\{e_1+e_j\mid 2\le j\le k\}\cup\{e_1-e_j\mid 2\le j\le k\}\cup\{e_1\}$$

And so

$$\begin{aligned}
\prod_{\alpha\in\left(\Delta^+_{\Sigma,i}\right)_{2k+1}} \langle\rho_K,\alpha\rangle &= \langle\rho_K,e_1\rangle \prod_{j=2}^{k} \langle\rho_K,e_1+e_j\rangle \langle\rho_K,e_1-e_j\rangle \\
&= \frac{1}{2}(2k-1)\prod_{j=2}^{k}(2k-j)(j-1) \\
&= \frac{1}{2}(2k-1)(2k-2)! \\
&= \frac{1}{2}(2k-1)! \\
&= \frac{1}{2}((2k+1)-2)!
\end{aligned}$$

And so we see that whether p is even or odd

$$\prod_{\alpha\in\left(\Delta^+_{\Sigma,i}\right)_{p}} \langle\rho_K,\alpha\rangle = \frac{1}{2}(p-2)!$$

Hence, for $SO(p,q)$

$$\prod_{\alpha\in\Delta^+_{\Sigma,i}} \langle\rho_K,\alpha\rangle = \frac{1}{4}(p-2)!(q-2)!$$

A.6. $G = Sp(n,\mathbb{C})$. $\qquad K = Sp_n \qquad \Sigma = C_n$

$$\begin{aligned}
\gamma_1 &= [2,0,\ldots,0] = 2e_1 \\
\gamma_2 &= [-2,2,0,\ldots,0] = 2e_2 \\
&\vdots \\
\gamma_n &= [0,\ldots,0,-2,2] = 2e_n
\end{aligned}$$

$$a_1\gamma_1+\cdots+a_n\gamma_n = [2a_1-2a_2,\ldots,2a_{n-1}-2a_n,2a_n]$$

$$\begin{aligned}
\rho_K &= ne_1+(n-1)e_2+\cdots+2e_{n-1}+e_n \\
\Rightarrow &\qquad \langle\rho_K,e_j\rangle = (n-j+1)
\end{aligned}$$

In this case we have $\gamma_j = 2e_j$, and so

$$\Delta^+_{\Sigma,i} = \{e_j\pm e_{j'} \mid 1\le j<j'\le i\}\cup\{e_j\pm e_{j'} \mid 1\le j\le i\ ,\ i+1<j'\le n\}\cup\{2e_j \mid 1\le j\le i\}$$

so

$$\begin{aligned}
\prod_{\alpha\in\Delta^+_{\Sigma,i}} \langle\rho_K,\alpha\rangle &= \left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{i}\langle\rho_K,e_j+e_{j'}\rangle\langle\rho_K,e_j-e_{j'}\rangle\right)\left(\prod_{j=1}^{i}\prod_{j'=i+1}^{n}\langle\rho_K,e_j+e_{j'}\rangle\langle\rho_K,e_j-e_{j'}\rangle\right)\\
&\quad\cdot\left(\prod_{j=1}^{i}\langle\rho_K,2e_j\rangle\right)\\
&= \left(\prod_{j=1}^{i-1}\prod_{j'=j+1}^{i}(2n-j-j'+2)(j'-j)\right)\left(\prod_{j=1}^{i}\prod_{j'=i+1}^{n}(2n-j-j'+2)(j'-j)\right)\\
&\quad\cdot\left(\prod_{j=1}^{i}(2n-2j+2)\right)\\
&= \left(\prod_{j=1}^{i}\frac{(2n+2-2j+1)!}{(2n+2-n-j-1)!}(n-j)!\right)\frac{2^i n!}{(n-i)!}\\
&= \frac{2^i n!}{(n-i)!}\left(\prod_{j=1}^{i}\frac{(2n-2j+3)!}{(n-j+1)}\right)\\
&= \frac{2^i n!}{(n-i)!}\left(\prod_{j=1}^{i}(2n-2j+3)!\right)\frac{(n-i)!}{n!}\\
&= 2^i\left(\prod_{j=1}^{i}(2n-2j+3)!\right)
\end{aligned}$$

A.7. $G = SL(2n,\mathbb{C})$. $\qquad K = SU_{2n} \qquad \Sigma = C_n$

$$\begin{aligned}
\gamma_1 &= [1,0,\ldots,0,1] = e_1 - e_{2n}\\
\gamma_2 &= [-1,1,0,\ldots,0,1,-1] = e_2 - e_{2n-1}\\
&\vdots\\
\gamma_n &= [0,\ldots,-1,2,-1,\ldots,0] = e_n - e_{n+1}
\end{aligned}$$

$$a_1\gamma_1+\cdots+a_n\gamma_n = [a_1-a_2,\ldots,a_{n-1}-a_n,2a_n,a_{n-1}-a_n,\ldots,a_1-a_2]$$

$$\begin{aligned}
\rho_K &= \frac{1}{2}\left((2n-1)e_1+(2n-3)e_2+\cdots-(2n-3)e_{2n-1}-(2n-1)e_{2n}\right)\\
&\Rightarrow \qquad \langle\rho_K,e_j\rangle = \frac{1}{2}(2n-2j+1)
\end{aligned}$$

$$\begin{aligned}
\Delta^+_{\Sigma,i} &= \{e_j-e_{j'} \mid 1\le j\le i \quad ; \quad j<j'\le 2n\}\\
&\quad \cup\{e_j-e_{j'} \mid i+1\le j\le 2n-i \quad ; \quad 2n-i<j'<n\}\\
&\quad \cup\{e_j-e_{j'} \mid 2n-i<j<j'\le 2n\}
\end{aligned}$$

$$
\begin{aligned}
\prod_{\alpha\in\Delta^+_{\Sigma,i}} \langle \rho_K,\alpha\rangle &= \left(\prod_{j=1}^{i}\prod_{j'=j+1}^{2n}\langle \rho_K, e_j - e_{j'}\rangle\right)\left(\prod_{j=i+1}^{2n-i}\prod_{j'=2n-i+1}^{2n}\langle \rho_K, e_j - e_{j'}\rangle\right)\left(\prod_{j=2n-i+1}^{2n-1}\prod_{j'=j+1}^{2n}\langle \rho_K, e_j - e_{j'}\rangle\right)\\
&= \left(\prod_{j=1}^{i}\prod_{j'=j+1}^{2n}\frac{1}{2}(j'-j)\right)\left(\prod_{j=i+1}^{2n-i}\prod_{j'=2n-i+1}^{2n}\frac{1}{2}(j'-j)\right)\left(\prod_{j=2n-i+1}^{2n-1}\prod_{j'=j+1}^{2n}\frac{1}{2}(j'-j)\right)\\
&= \left(\prod_{j=1}^{i}\left(\frac{1}{2}\right)^{2n-j}\prod_{j'=j+1}^{2n}(j'-j)\right)\left(\prod_{j=i+1}^{2n-i}\left(\frac{1}{2}\right)^{i}\prod_{j'=2n-i+1}^{2n}(j'-j)\right)\\
&\quad\cdot\left(\prod_{j=2n-i+1}^{2n-1}\left(\frac{1}{2}\right)^{2n-j}\prod_{j'=j+1}^{2n}(j'-j)\right)\\
&= \left(\prod_{j=1}^{i}\left(\frac{1}{2}\right)^{2n-j}(2n-j)!\right)\left(\prod_{j=i+1}^{2n-i}\left(\frac{1}{2}\right)^{i}\frac{(2n-j)!}{(2n-i-j)!}\right)\left(\prod_{j=2n-i+1}^{2n-1}\left(\frac{1}{2}\right)^{2n-j}(2n-j)!\right)\\
&= \left(\frac{1}{2}\right)^{2ni-i(i+1)/2}\left(\frac{1}{2}\right)^{i(2n-2i)}\left(\frac{1}{2}\right)^{i(i-1)/2}\left(\prod_{j=1}^{i}(2n-j)!\right)\left(\prod_{j=i+1}^{2n-i}\frac{(2n-j)!}{(2n-i-j)!}\right)\\
&\quad\cdot\left(\prod_{j=2n-i+1}^{2n-1}(2n-j)!\right)\\
&= \left(\frac{1}{2}\right)^{i(4n-2i-1)}\left(\prod_{j=1}^{2n-1}(2n-j)!\right)\left(\prod_{j=1}^{2n-2i}\frac{1}{(2n-2i-j)!}\right)
\end{aligned}
$$

A.8. $G = SO(4n,\mathbb{C})$. $K = SO_{4n}$ $\Sigma = C_n$

$$
\begin{aligned}
\gamma_1 &= [0,1,0,\dots,0] = e_1 + e_2\\
\gamma_2 &= [0,-1,0,1,0,\dots,0] = e_3 + e_4\\
\gamma_3 &= [0,0,0,-1,0,1,0,\dots,0] = e_5 + e_6\\
&\;\;\vdots\\
\gamma_{n-1} &= [0,\dots,0,-1,0,1,0,0] = e_{2n-3} + e_{2n-2}\\
\gamma_n &= [0,\dots,0,0,-1,0,2] = e_{2n-1} + e_{2n}
\end{aligned}
$$

$$
a_1\gamma_1 + \cdots + a_n\gamma_n = [0, a_1 - a_2, 0, a_2 - a_3, \dots, 0, a_{n-1} - a_n, 0, 2a_n]
$$

$$
\begin{aligned}
\rho_K &= (2n-1)\,e_1 + (2n-2)\,e_2 + \cdots + e_{2n-1}\\
&\Rightarrow \quad \langle \rho_K, e_j\rangle = 2n - j
\end{aligned}
$$

$$
\begin{aligned}
\Delta^+_{\Sigma,i} &= \{e_j + e_{j'} \mid 1 \le j \le 2i \quad , \quad j < j' \le 2n\}\\
&\quad \cup \{e_{2j-1} - e_{j'} \mid 1 \le j \le i \quad , \quad 2j < j' \le 2n\}\\
&\quad \cup \{e_{2j} - e_{j'} \mid 1 \le j \le i \quad , \quad 2j < j' \le 2n\}
\end{aligned}
$$

$$\begin{aligned}
\prod_{\alpha\in\Delta^+_{\Sigma,i}} \langle\rho_K,\alpha\rangle &= \left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}\langle\rho_K,e_j+e_{j'}\rangle\right)\left(\prod_{j=1}^{i}\prod_{j'=2j+1}^{2n}\langle\rho_K,e_{2j-1}-e_{j'}\rangle\right)\left(\prod_{j=1}^{i-1}\prod_{j'=2j+1}^{2n}\langle\rho_K,e_{2j}-e_{j'}\rangle\right)\\
&= \left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}(4n-j-j')\right)\left(\prod_{j=1}^{i}\prod_{j'=2j+1}^{2n}(j'-2j+1)\right)\left(\prod_{j=1}^{i-1}\prod_{j'=2j+1}^{2n}(j'-2j)\right)\\
&= \left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}(4n-j-j')\right)\left(\prod_{j=1}^{i}\prod_{j'=2j}^{2n}(j'-2j+1)\right)\left(\prod_{j=1}^{i-1}\prod_{j'=2j+1}^{2n}(j'-2j)\right)\\
&= \left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}(4n-j-j')\right)\left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}(j'-j)\right)\\
&= \left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}(4n-j-j')\,(j-j')\right)
\end{aligned}$$

Adopting Formula (D) to the case at hand ($i\to 2i,\ n\to 2n,\ N\to 2n$)

$$\left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}(4n-j-j')\,(j'-j)\right)=\prod_{j=1}^{2i}\left(\frac{(4n-2j+1)!}{(4n-2n-j-1)!}\,(2n-j)!\right)=\prod_{j=1}^{2i}\left(\frac{(4n-2j+1)!}{(2n-j-1)!}\,(2n-j)!\right)$$

we conclude

$$\prod_{\alpha\in\Delta^+_{\Sigma,i}}\langle\rho_K,\alpha\rangle=\prod_{j=1}^{2i}\left(\frac{(4n-2j+1)!}{(2n-j-1)!}\,(2n-j)!\right)$$

A.9. $G=E_7(\mathbb{C})$. $\quad K=E_7 \quad \Sigma=C_3$

$$\begin{aligned}
\gamma_1 &= [1,0,0,0,0,0,0]=-e_7+e_8\\
\gamma_2 &= [-1,0,0,0,0,1,0]=e_5+e_6\\
\gamma_3 &= [0,0,0,0,0,0,-1,2]=-e_5+e_6
\end{aligned}$$

$$a_1\gamma_1+a_2\gamma_2+a_3\gamma_3==[a_1-a_2,0,0,0,0,0,a_2-a_3,2a_3]$$

We carried out the calculation of $\prod_{\alpha\in\Delta^+_{\Sigma,1}}\langle\rho_K,\alpha\rangle$ for $i=1,2,3$ using Maple, and obtained the following results.

$$\begin{aligned}
\prod_{\alpha\in\Delta^+_{\Sigma,1}}\langle\rho_K,\alpha\rangle &= 2^{29}3^{13}5^{6}7^{4}11^{3}13^{2}17\\
\prod_{\alpha\in\Delta^+_{\Sigma,2}}\langle\rho_K,\alpha\rangle &= 2^{42}3^{19}5^{9}7^{6}11^{3}13^{2}17\\
\prod_{\alpha\in\Delta^+_{\Sigma,3}}\langle\rho_K,\alpha\rangle &= 2^{42}3^{19}5^{9}7^{6}11^{3}13^{2}17
\end{aligned}$$

A.10. $G=SO(p,\mathbb{C})$. $\quad K=SO_p \quad \Sigma=C_2$

$$\begin{aligned}
\gamma_1 &= [0,1,0,\dots,0]=e_1+e_2\\
\gamma_2 &= [0,-1,0,1,0,\dots,0]=e_3+e_4
\end{aligned}$$

$$a_1\gamma_1+a_2\gamma_2=[0,a_1-a_2,0,a_2,0,\dots,0]$$

$p = 2k$.

$$\begin{aligned} \rho_K &= (k-1)\,e_1 + (k-2)\,e_2 + \cdots + e_{k-1} \\ &\Rightarrow \quad \langle \rho_K, e_j \rangle = k - j \end{aligned}$$

$$\begin{aligned} \Delta^+_{\Sigma,i} &= \{e_j + e_{j'} \mid 1 \le j \le 2i \quad , \quad j < j' \le k\} \\ &\cup \{e_{2j-1} - e_{j'} \mid 1 \le j \le i \quad , \quad 2j < j' \le k\} \\ &\cup \{e_{2j} - e_{j'} \mid 1 \le j \le i \quad , \quad 2j < j' \le k\} \end{aligned}$$

$$\begin{aligned} \prod_{\alpha \in \Delta^+_{\Sigma,i}} \langle \rho_K, \alpha \rangle &= \left(\prod_{j=1}^{2i} \prod_{j'=j+1}^{k} \langle \rho_K, e_j + e_{j'} \rangle \right) \left(\prod_{j=1}^{i} \prod_{j'=2j+1}^{k} \langle \rho_K, e_{2j-1} - e_{j'} \rangle \right) \left(\prod_{j=1}^{i} \prod_{j'=2j+1}^{k} \langle \rho_K, e_{2j} - e_{j'} \rangle \right) \\ &= \left(\prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (2k - j - j') \right) \left(\prod_{j=1}^{i} \prod_{j'=2j+1}^{k} (j' - 2j + 1) \right) \left(\prod_{j=1}^{i} \prod_{j'=2j+1}^{k} (j' - 2j) \right) \\ &= \left(\prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (2k - j - j') \right) \left(\prod_{j=1}^{i} \prod_{j'=2j}^{k} (j' - 2j + 1) \right) \left(\prod_{j=1}^{i} \prod_{j'=2j+1}^{k} (j' - 2j) \right) \\ &= \left(\prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (2k - j - j') \right) \left(\prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (j' - j) \right) \\ &= \prod_{j=1}^{2i} \prod_{j'=j+1}^{k} ((2k - j - j')\,(j' - j)) \\ &= \prod_{j=1}^{2i} \left(\frac{(2k - 2j + 1)!}{(k - j - 1)!}\,(k - j)! \right) \\ &= \prod_{j=1}^{2i} (2k - 2j + 1)!\,(k - j) \\ &= \left(\frac{1}{2} \right)^{2i} \left(\prod_{j=1}^{2i} (2k - 2j + 1)!\,(2k - 2j) \right) \end{aligned}$$

$p = 2k + 1$.

$$\begin{aligned} \rho_K &= \left(k - \frac{1}{2} \right) e_1 + \left(k - \frac{3}{2} \right) e_2 + \cdots + \frac{1}{2} e_k \\ &\Rightarrow \quad \langle \rho_K, e_j \rangle = \frac{1}{2}\,(2k - 2j + 1) \end{aligned}$$

$$\begin{aligned} \Delta^+_{\Sigma,i} &= \{e_j + e_{j'} \mid 1 \le j \le 2i \quad , \quad j < j' \le k\} \\ &\cup \{e_{2j-1} - e_{j'} \mid 1 \le j \le i \quad , \quad 2j < j' \le k\} \\ &\cup \{e_{2j} - e_{j'} \mid 1 \le j \le i \quad , \quad 2j < j' \le k\} \\ &\cup \{e_j \mid 1 \le j \le 2i\} \end{aligned}$$

$$\begin{aligned}
\prod_{\alpha\in\Delta^+_{\Sigma,i}}\langle\rho_K,\alpha\rangle &= \left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{k}\langle\rho_K,e_j+e_{j'}\rangle\right)\left(\prod_{j=1}^{i}\prod_{j'=2j+1}^{k}\langle\rho_K,e_{2j-1}-e_{j'}\rangle\right)\left(\prod_{j=1}^{i-1}\prod_{j'=2j+1}^{k}\langle\rho_K,e_{2j}-e_{j'}\rangle\right)\\
&\quad\cdot\left(\prod_{j=1}^{2i}\langle\rho_K,e_j\rangle\right)\\
&= \left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{k}(2k-j-j'+1)\right)\left(\prod_{j=1}^{i}\prod_{j'=2j+1}^{k}(j'-2j+1)\right)\left(\prod_{j=1}^{i-1}\prod_{j'=2j+1}^{k}(j'-2j)\right)\\
&\quad\cdot\left(\prod_{j=1}^{2i}\frac{1}{2}(2k-2j+1)\right)\\
&= \left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{k}(2k-j-j'+1)\right)\left(\prod_{j=1}^{i}\prod_{j'=2j}^{k}(j'-2j+1)\right)\left(\prod_{j=1}^{i-1}\prod_{j'=2j+1}^{k}(j'-2j)\right)\\
&\quad\cdot\left(\frac{1}{2}\right)^{2i}\left(\prod_{j=1}^{2i}(2k-2j+1)\right)\\
&= \left(\frac{1}{2}\right)^{2i}\left(\prod_{j=1}^{2i}(2k-2j+1)\right)\left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{k}(2k-j-j'+1)\right)\left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{k}(j'-j)\right)\\
&= \left(\frac{1}{2}\right)^{2i}\left(\prod_{j=1}^{2i}(2k-2j+1)\right)\left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{k}(2k-j-j'+1)(j-j')\right)\\
&= \left(\frac{1}{2}\right)^{2i}\left(\prod_{j=1}^{2i}(2k-2j+1)\right)\left(\prod_{j=1}^{2i}\frac{(2k-2j+2)!}{(k-j)!}(k-j)!\right)\\
&= \left(\frac{1}{2}\right)^{2i}\left(\prod_{j=1}^{2i}(2k-2j+2)!\,(2k-2j+1)\right)
\end{aligned}$$

Thus, we see again we can consolidate the formulas for $p=2k$ and $p=2k+1$.

$$\prod_{\alpha\in\Delta^+_{\Sigma,i}}\langle\rho_K,\alpha\rangle=\prod_{j=1}^{2i}\frac{1}{2}(p-2j)(p-2j+1)!$$

A.11. $G=Sp(n,n)$. $\quad K=Sp_n\times Sp_n \quad \Sigma=C_n$

$$\begin{aligned}
\gamma_1 &= [1,0,\dots,0;1,0,\dots,0]=e_1+\tilde{e}_1\\
\gamma_2 &= [-1,1,0,\dots,0;-1,1,0,\dots,0]=e_2+\tilde{e}_2\\
&\ \vdots\\
\gamma_n &= [0,\dots,0,-1,1;0,\dots,0,-1,1]=e_n+\tilde{e}_n
\end{aligned}$$

$$a_1\gamma_1+\dots+a_n\gamma_n=[a_1-a_2,\dots,a_{n-1}-a_n,a_n;a_1-a_2,\dots,a_{n-1}-a_n]$$

$$\begin{aligned}
\rho_K &= ne_1+(n-1)e_2+\dots+2e_{n-1}+e_n+n\tilde{e}_1+(n-1)\tilde{e}_2+\dots+2\tilde{e}_{n-1}+\tilde{e}_n\\
&\Rightarrow \quad \langle\rho_K,e_j\rangle=n-j+1 \quad , \quad \langle\rho_K,\tilde{e}_j\rangle=n-j+1
\end{aligned}$$

$$\begin{aligned}
\Delta^+_{\Sigma,i} &= \{e_j+e_{j'}\mid 1\le j\le i\ ,\ j<j'\le n\}\cup\{e_j-e_{j'}\mid 1\le j\le i\ ,\ j<j'\le n\}\cup\{2e_j\mid 1\le j\le i\}\\
&\quad\cup\{\tilde{e}_j+\tilde{e}_{j'}\mid 1\le j\le i\ ,\ j<j'\le n\}\cup\{\tilde{e}_j-\tilde{e}_{j'}\mid 1\le j\le i\ ,\ j<j'\le n\}\cup\{2\tilde{e}_j\mid 1\le j\le i\}
\end{aligned}$$

$$
\begin{aligned}
\prod_{\alpha\in\Delta^+_{\Sigma,i}} \langle \rho_K, \alpha\rangle &= \left(\prod_{j=1}^{i}\prod_{j'=j+1}^{n}\langle \rho_K, e_j + e_{j'}\rangle\right)\left(\prod_{j=1}^{i}\prod_{j'=j+1}^{n}\langle \rho_K, e_j - e_{j'}\rangle\right)\left(\prod_{j=1}^{i}\langle \rho_K, 2e_j\rangle\right)\\
&\quad \cdot\left(\prod_{j=1}^{i}\prod_{j'=j+1}^{n}\langle \rho_K, \tilde{e}_j + \tilde{e}_{j'}\rangle\right)\left(\prod_{j=1}^{i}\prod_{j'=j+1}^{n}\langle \rho_K, \tilde{e}_j - \tilde{e}_{j'}\rangle\right)\left(\prod_{j=1}^{i}\langle \rho_K, 2\tilde{e}_j\rangle\right)\\
&= \left(\prod_{j=1}^{i}\prod_{j'=j+1}^{n}(2n-j-j'+2)\,(j'-j)\right)^2\left(\prod_{j=1}^{i}2\,(n-j+1)\right)^2\\
&= 2^{2i}\left(\frac{(n)!}{(n-i)!}\right)^2\left(\prod_{j=1}^{i}\left(\frac{(2n-2j+3)!}{(n-j+1)!}\,(n-j)!\right)\right)^2\\
&= 2^{2i}\left(\frac{(n)!}{(n-i)!}\right)^2\left(\prod_{j=1}^{i}\left(\frac{(2n-2j+3)!}{n-j+1}\right)\right)^2\\
&= 2^{2i}\left(\frac{(n)!}{(n-i)!}\right)^2\left(\frac{(n-i)!}{n!}\prod_{j=1}^{i}(2n-2j+3)!\right)^2\\
&= 2^{2i}\left(\prod_{j=1}^{i}(2n-2j+3)!\right)^2
\end{aligned}
$$

A.12. $G = SL\,(2n, \mathbb{H})$. $\quad K = Sp_{2n} \quad \Sigma = C_n$

$$
\begin{aligned}
\gamma_1 &= [0,1,0,\ldots,0] = e_1 + e_2\\
\gamma_2 &= [0,-1,0,1,0,\ldots,0] = e_3 + e_4\\
\gamma_3 &= [0,0,0,-1,0,1,0,\ldots,0] = e_5 + e_6\\
&\ \vdots\\
\gamma_n &= [0,\ldots,0,-1.0,1] = e_{2n-1} + e_{2n}
\end{aligned}
$$

$$
a_1\gamma_1 + \cdots + a_n\gamma_n = [0, a_1 - a_2, 0, a_2 - a_3, \ldots, 0, a_{n-1} - a_n, 0, a_n]
$$

$$
\begin{aligned}
\rho_K &= 2ne_1 + (2n-1)\,e_2 + \cdots + 2e_{2n-1} + e_{2n}\\
&\Rightarrow \quad \langle \rho_K, e_j\rangle = 2n - j + 1
\end{aligned}
$$

$$
\begin{aligned}
\Delta^+_{\Sigma,i} = \ & \{e_j + e_{j'} \mid 1 \le j \le 2i \quad , \quad j < j' \le 2n\}\\
& \cup \{e_{2j-1} - e_{j'} \mid 1 \le j \le i \quad , \quad 2j < j' \le 2n\}\\
& \cup \{e_{2j} - e_{j'} \mid 1 \le j \le i \quad , \quad 2j < j' \le 2n\}\\
& \cup \{2e_j \mid 1 \le j \le 2i\}
\end{aligned}
$$

$$
\begin{aligned}
\prod_{\alpha\in\Delta^+_{\Sigma,i}}\langle\rho_K,\alpha\rangle &= \left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}\langle\rho_K,e_j+e_{j'}\rangle\right)\left(\prod_{j=1}^{i}\prod_{j'=2j+1}^{2n}\langle\rho_K,e_{2j-1}-e_{j'}\rangle\right)\left(\prod_{j=1}^{i-1}\prod_{j'=2j+1}^{2n}\langle\rho_K,e_{2j}-e_{j'}\rangle\right)\\
&\quad\cdot\left(\prod_{j=1}^{2i}\langle\rho_K,2e_j\rangle\right)\\
&= \left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}(4n-j-j'+2)\right)\left(\prod_{j=1}^{i}\prod_{j'=2j+1}^{2n}(j'-2j+1)\right)\left(\prod_{j=1}^{i-1}\prod_{j'=2j+1}^{2n}(j'-2j)\right)\\
&\quad\cdot\left(\prod_{j=1}^{2i}2\,(2n-j+1)\right)\\
&= \left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}(4n-j-j'+2)\right)\left(\prod_{j=1}^{i}\prod_{j'=2j}^{2n}(j'-2j+1)\right)\left(\prod_{j=1}^{i-1}\prod_{j'=2j+1}^{2n}(j'-2j)\right)\\
&\quad\cdot 2^{2i}\frac{(2n)!}{(2n-2i)!}\\
&= 2^{2i}\frac{(2n)!}{(2n-2i)!}\left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}(4n-j-j'+2)\right)\left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}(j'-j)\right)\\
&= 2^{2i}\frac{(2n)!}{(2n-2i)!}\left(\prod_{j=1}^{2i}\prod_{j'=j+1}^{2n}(4n-j-j'+2)\,(j-j')\right)\\
&= 2^{2i}\frac{(2n)!}{(2n-2i)!}\left(\prod_{j=1}^{2i}\left(\frac{(4n-2j+3)!}{(2n-j+1)!}\,(n-j)!\right)\right)
\end{aligned}
$$

where in the last step we employed Formula (D).

$$
\left(\prod_{j=1}^{i}\prod_{j'=j+1}^{n}(N-j-j')\,(j'-j)\right)=\prod_{j=1}^{i}\left(\frac{(N-2j+1)!}{(N-n-j-1)!}\,(n-j)!\right)
$$

A.13. $G=SO\,(p,1)$. $\qquad K=SO\,(p)\qquad \Sigma=C_1$

$$\gamma_1=[1,0,\ldots,0]=e_1$$

$$a_1\gamma_1=[a_1,0,\ldots,0]$$

A.13.1. *Case A.* $p=2k$.

$$
\begin{aligned}
\rho_K &= (k-1)\,e_1+(k-2)\,e_2+\cdots+e_{k-1}\\
&\Rightarrow \quad \langle\rho_K,e_j\rangle=k-j
\end{aligned}
$$

$$\Delta^+_{\Sigma,1}=\{e_1+e_j\mid j=2,\ldots,k\}\cup\{e_1-e_j\mid j=2,\ldots,k\}$$

$$\begin{aligned}
\prod_{\alpha\in\Delta^+_{\Sigma,i}} \langle\rho_K,\alpha\rangle &= \prod_{j=2}^{k}(2k-j-1)(j-1) \\
&= (2k-3)(1)(2k-4)(2)\cdots(k)(k-2)(k-1)(k-1) \\
&= (k-1)(2k-3)! \\
&= \frac{1}{2}(2k-2)(2k-3)! \\
&= \frac{1}{2}(2k-2)!
\end{aligned}$$

A.13.2. *Case B.* $p = 2k+1$.

$$\rho_K = \frac{1}{2}\left((2k-1)e_1 + (2k-3)e_2 + \cdots + 3e_{k-1} + e_k\right)$$

$$\Delta^+_{\Sigma,1} = \{e_1\} \cup \{e_1 + e_j \mid 1 < j \leq k\} \cup \{e_1 - e_j \mid 1 < j \leq k\}$$

$$\begin{aligned}
\prod_{\alpha\in\Delta^+_{\Sigma,i}} \langle\rho_K,\alpha\rangle &= \langle\rho_K,e_1\rangle \prod_{j=2}^{k}\langle\rho_K,e_1+e_j\rangle\langle\rho_K,e_1-e_j\rangle \\
&= \frac{1}{2}(2k-1)\prod_{j=2}^{k}(2k-j)(j-1) \\
&= \frac{2k-1}{2}(2k-2)(1)(2k-3)(2)\cdots(k)(k-1) \\
&= \frac{1}{2}(2k-1)! \\
&= \frac{1}{2}((2k+1)-2)!
\end{aligned}$$

Thus, whether p is even or odd we have

$$\prod_{\alpha\in\Delta^+_{\Sigma,i}} \langle\rho_K,\alpha\rangle = \frac{1}{2}(p-2)!$$